Problems in Quantum Field Theory

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1 Classical Electrodynamics

1. How much is 1 Tesla in natural units?
2. Write the field-strength tensor $F^{\mu\nu}$ as a function of the electric field E_i and of the magnetic field B_i.
3. What is the field of a static charge in each of the following gauges:

$$\begin{aligned} &\text{Lorenz} \quad \partial_\mu A^\mu = 0, \\ &\text{Axial} \qquad n_\mu A^\mu = 0, \\ &\text{Coulomb} \quad \vec{\nabla}.\vec{A} = 0? \end{aligned}$$

4. Show that the rapidity y is additive with respect to boosts and that $y \sim v$ if $c \to \infty$.

2 The Principle of Extremal Action

1. Derive the canonical equation of motion from the Hamiltonian

$$H(p,q) = \sum_i [p_i \dot{q}_i(q,p) - L(q, \dot{q}(q,p))].$$

2. Find, for a single particle, the action which is invariant with respect to boosts and obtain the Euler-Lagrange equation, the momentum and the energy.
3. Show that, with fields, the canonical equations of motion are given by

$$\frac{\partial H}{\partial \pi} = \dot{\varphi}, \qquad \frac{\partial H}{\partial \varphi} = -\dot{\pi}.$$

4. Show that $F_{\mu\nu}\tilde{F}^{\mu\nu}$ is a total derivative.

3 Quantization

1. Show that the current

$$j_\mu = -\frac{i}{2}(\varphi \partial_\mu \varphi^* - \varphi^* \partial_\mu \varphi),$$

is conserved if φ obeys the Klein-Gordon equation.

2. Using the two-component wave function

$$\begin{pmatrix} \theta \\ \chi \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \varphi + \frac{i}{m}\frac{\partial \varphi}{\partial t} \\ \varphi - \frac{i}{m}\frac{\partial \varphi}{\partial t} \end{pmatrix},$$

instead of φ, rewrite the Klein-Gordon equation in the Schrödinger form.

3. Show that the Dirac and the Weyl representations are equivalent.

4. Show that, for fermions, helicity and angular momentum are conserved.

5. Let us introduce the Weyl fields

$$\psi_L = \frac{1}{2}(1 - \gamma_5)\psi,$$
$$\psi_R = \frac{1}{2}(1 + \gamma_5)\psi,$$

where ψ is a Dirac spinor. Derive the equations of motion for these fields. Show that they are decoupled in the case of a massless spinor.

6. Prove that if $\psi(x)$ is a solution of the Dirac equation, it is also a solution of the Klein-Gordon equation.

7. Prove that the quantity $\bar{\psi}(x)\gamma_\mu \partial^\mu \psi(x)$ is a Lorentz scalar. Find its transformation rules under the discrete transformations.

8. Show that $S_p = \gamma_0$.

9. Find the matrices C and P in the Weyl representation of the γ-matrices.

10. Compute the commutators $[\phi(x), \phi(x')]_{t=t'}$ and $[\pi(x), \phi(x')]_{t=t'}$.

4 Free Fields

1. Prove that the equations of motion remain unchanged if the divergence of an arbitrary field is added to the Lagrangian density.

2. The Lagrangian density for a massive vector field A^μ is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu.$$

Prove that the equation $\partial_\mu A^\mu = 0$ is a consequence of the equations of motion.

3. For photons, compute the commutator $[a^\lambda, a^{\lambda' +}]$ from the canonical field commutations.

4. Show, in Feynman gauge, that

$$H = \int \frac{d^3k}{(2\pi)^3} E_k \sum_{\lambda=1}^{3} (a^{\lambda +} a^{\lambda} - a^{0+} a^0).$$

5 Interacting Fields

1. Interpret the following Lagrangian densities and find the corresponding Euler-Lagrange equations

   - $\mathcal{L} = -(\partial_\mu A^\nu)(\partial_\nu A^\mu) + \frac{1}{2} m^2 A_\mu A^\mu + \frac{\lambda}{2}(\partial_\mu A^\mu)^2$,
   - $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu$,
   - $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 + \frac{1}{4}\lambda \phi^4$,
   - $\mathcal{L} = (\partial_\mu \phi - ieA_\mu \phi)(\partial^\mu \phi^* + ieA^\mu \phi^*) - m^2 \phi^* \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$,
   - $\mathcal{L} = \bar{\psi}(i\gamma_\mu \partial^\mu - m)\psi + \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 + \frac{1}{4}\lambda \phi^4 - ig\bar{\psi}\gamma_5 \psi \phi$.

2. Derive the formula for the cross section $d\sigma$ from the S-matrix.

3. Prove that $D_F(x - x')$ is a solution of $(\Box + m^2)D = -i\delta^{(4)}(x - y)$.

4. Calculate the photon propagator $D_{\mu\nu}(k)$ in a general covariant gauge.

6 QED Cross section

1. Prove that the sum over polarization gives

$$\sum_s u^s(p)\bar{u}^s(p) = \gamma.p + m, \qquad \sum_s v^s(p)\bar{v}^s(p) = \gamma.p - m.$$

2. Check that $\gamma_0^+ = \gamma_0$ and that $\gamma_0 \gamma_\mu \gamma_0 = \gamma_\mu^+$.

3. Compute $Tr(\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta)$ and $\gamma_\mu \gamma^\alpha \gamma^\beta \gamma^\mu$.

4. What are the Feynman rules for the Lagrangians of exercice 5.1?

5. Show that $s + t + u = \sum m^2$.

6. Calculate the invariant amplitude of Bhabha scattering, $e^- e^+ \to e^- e^+$ via Wick's theorem and directly from Feynman rules.

7. Compute the invariant amplitude of $e^+ e^- \to \mu^+ \mu^-$.

8. Compute the Compton scattering cross section.

9. Calculate the invariant amplitude of Møller scattering.

10. Compute the invariant amplitude of $e^+ e^- \to \mu^+ \mu^-$ via a massive spin-1 boson exchange.