4. FREE FIELDS

- COMPLEX SCALAR FIELD
- DIRAC FIELD
- GAUGE FIELD

EXAMPLE : COMPLEX ϕ

$$\mathcal{L} = \mathcal{L}(\phi_1) + \mathcal{L}(\phi_2)$$

CHANGE FIELD DEFINITION

$$\phi = \phi_1 + i\,\phi_2$$

$$\mathcal{L} = \partial_\mu \phi\, \partial^\mu \phi^* - m^2 \phi\phi^*$$

$$\pi = \dot{\phi}^* \qquad \pi^* = \dot{\phi}$$

$$\phi = \int \frac{d^3p}{\sqrt{2p_0}\,(2\pi)^3} \left(a_{\vec{p}}\, e^{i\vec{p}.\vec{x}} + b^\dagger_{\vec{p}}\, e^{-i\vec{p}.\vec{x}} \right)$$

$$\phi^* = \quad \cdot \quad \left(a^\dagger_{\vec{p}}\, e^{-i\vec{p}.\vec{x}} + b_{\vec{p}}\, e^{i\vec{p}.\vec{x}} \right)$$

$$\mathcal{H} = \int d^3x \left(\pi \dot{\phi} + \pi^* \dot{\phi}^* - \mathcal{L} \right)$$

$$= \int d^3x \left(\pi\pi^* + \vec{\nabla}\phi . \vec{\nabla}\phi^* + m^2 \phi\phi^* \right)$$

$$\cdots$$

$$= \int \frac{d^3p}{(2\pi)^3}\, E_p \left(a^\dagger_{\vec{p}}\, a_{\vec{p}} + b^\dagger_{\vec{p}}\, b_{\vec{p}} \right) + C$$

$$j^\mu = i\left[(\partial^\mu \phi^*)\,\phi - \phi^* \partial^\mu \phi \right]$$

$$Q = \int d^3x\, j^0 = \int \frac{d^3p}{(2\pi)^3 (2p_0)} \big(\underset{\underset{Q>0}{\uparrow}}{a^\dagger_{\vec{p}}}\, a_{\vec{p}} - \underset{\underset{Q<0}{\uparrow}}{b^\dagger_p}\, b_p \big)$$

NORMAL ORDERING

$$H = :H: + C$$

$::$ $\left\{ \begin{array}{l} \text{CREATION TO THE LEFT} \\ \text{DESTRUCTION TO THE RIGHT} \end{array} \right.$

HEISENBERG PICTURE:

$\left. \begin{array}{l} \psi \quad \text{FIXED} \\ i\dot{O} = -[H, O] \end{array} \right\}$ C DOES NOT MATTER!

SCHRÖDINGER PICTURE

$\psi_s = e^{-Ht}\psi_H$ $\longrightarrow$ ψ_s DOES NOT EXIST!

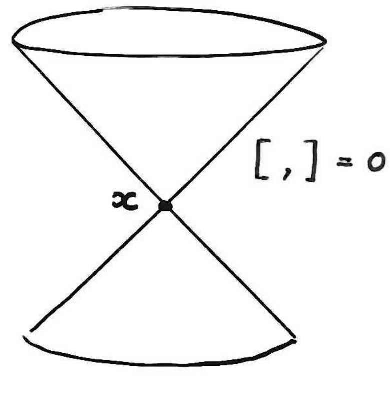
$O_s = e^{-iHt} O_H e^{iHt}$

$\Rightarrow$ IN ORDER TO WORK IN EITHER PICTURE, NORMAL-ORDER ALL OPERATORS

CAUSALITY

$$[\phi(x), \phi^*(y)] = \int \frac{d^3p}{2(2\pi)^3 p^0} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(y-x)} \right)$$

$$= D(x-y) - D(y-x)$$



$(x-y)^2 < 0$: in the frame $x^0 - y^0 = 0$, rotate $\vec{x} - \vec{y}$ into $\vec{y} - \vec{x}$

$$\Rightarrow [\phi(x), \phi^*(y)] = 0 \quad \text{if} \quad x^2 + y^2 - 2xy < 0$$

$$\underbrace{\langle 0|\phi(x)\phi(x)^*|0\rangle}_{\text{particle}} - \underbrace{\langle 0|\phi^*(x)\phi(x)|0\rangle}_{\text{antiparticle}}$$

PROBLEM 1 : DIRAC FIELD

$$\mathcal{L} = \bar{\psi}\,(i\gamma\cdot\partial - m)\,\psi$$
$$\pi = i\psi^{\dagger}$$
$$H = \int d^3x\; \psi^{\dagger}(-i\vec{\nabla}\cdot\vec{\alpha} - m\beta)\,\psi$$

$$[\psi_a(\vec{x}), \psi_b^{\dagger}(y)] = \delta^{(3)}(\vec{x}-\vec{y})\,\delta_{ab}$$

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \sum_s \left(a_{\vec{p}}^s u_s e^{i\vec{p}\cdot\vec{x}} + b_{\vec{p}}^{s\dagger} v_s e^{-i\vec{p}\cdot\vec{x}}\right)$$

→ WRONG ALGEBRA:

$$[a^r, a^{s\dagger}] = [b^{\dagger r}, b^s] = (2\pi)^3\,\delta^{(3)}\,\delta_{rs}$$

→ $b \leftrightarrow b^{\dagger}$

H NOT POSITIVE DEFINITE

$$H = \int \frac{d^3p}{(2\pi)^3} \sum E_p\,(a^{\dagger}a - b^{\dagger}b)$$

$$[\psi(x), \bar{\psi}(x)] = \langle 0|\psi(x)\bar{\psi}(y)|0\rangle$$

AS $\psi(x)|0\rangle = 0$

→ NEED $E<0$ TO HAVE CAUSALITY!

SOLUTION

CHANGE THE ALGEBRAS:

$$[\;,\;] \longrightarrow \{\;,\;\}$$

$$\{\psi_a(x), \psi_b^{\dagger}(y)\}_{ET} = \delta^{(3)}(\vec{x}-\vec{y})\,\delta^{ab}$$
$$\{\psi, \psi\}_{ET} = \{\psi^{\dagger}, \psi^{\dagger}\}_{ET} = 0$$

$$\psi = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \sum a_{\vec{p}}^s u^s(\vec{p})\, e^{-ip\cdot x} + b_{\vec{p}}^{s\dagger} v^s(\vec{p})\, e^{ip\cdot x}$$

$$\bar{\psi} = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \sum b_{\vec{p}}^s \bar{v}^s e^{-ip\cdot x} + a_{\vec{p}}^{s\dagger} \bar{u}^s(\vec{p})\, e^{ip\cdot x}$$

$$\Rightarrow \left[\begin{array}{l} \{a_{\vec{p}}^r, a_{\vec{q}}^{s\dagger}\} = \{b_{\vec{p}}^s, b_{\vec{q}}^{s\dagger}\} = (2\pi)^3\,\delta^{(3)}(\vec{p}-\vec{q})\,\delta_{rs} \\ \{a^r, a^s\} = \{b^r, b^s\} = 0 \\ \{a_s^{\dagger}, a_r^{\dagger}\} = \{b_s^{\dagger}, b_r^{\dagger}\} = 0 \end{array}\right.$$

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s E_p\,(a_p^{s\dagger} a_p^s + b_p^{s\dagger} b_p^s)$$

CONSEQUENCES

$\{a_r^+, a_s^+\} = 0 \quad \Rightarrow \quad 1 \propto 0$

PARTICLE / STATE

$\{a^+, a\} = \{a, a^+\}$

SYMETRY $\quad a \leftrightarrow a^+$

hole $\quad$ particle

OBSERVABLES MUST CONTAIN BILENEARS

$\{\psi(x), \psi(y)\}$ VANISHES IF $(x-y)^2 < 0$

BUT WE WANT $[O(x), O(y)] = 0$

$[ab, cd] = a\{bc\}d - ac\{bd\} + \{ac\}db - c\{ad\}b$

SPIN-STATISTICS THEOREM :

$E > 0$, CAUSALITY

$\Rightarrow \begin{cases} \text{spin } \frac{1}{2}+n & \leftrightarrow \text{ fermions} \\ \text{spin } n & \leftrightarrow \text{ bosons} \end{cases}$

PROBLEM 2 : GAUGE BOSONS

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} = -\frac{1}{4} (\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

$\rightarrow \quad \pi^0 = 0$

$[A^0, \pi^0] \sim \delta^{(3)} \quad ??$

$\rightarrow$ • THROW A^0 AWAY AND KEEP ONLY $\vec{A}$

e.g. $A^0 = 0$ AS GAUGE

• CHANGE $\mathcal{L}$ + RESTRICT Hilbert sp.

$$\boxed{\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{\lambda}{2} (\partial_\mu A^\mu)^2}$$

$$\Rightarrow \begin{cases} \Box A^\mu - (1-\lambda)\, \partial^\mu (\partial \cdot A) = 0 \\ \pi^\mu = F^{\mu 0} - \lambda g^{\mu 0} (\partial . A) \end{cases}$$

$[A_\rho, A_\nu] = [\pi_\rho, \pi_\nu] = 0$

$\lambda = 1$ (Feynman)

$\pi_0 = -\partial . A = -\dot{A}_0 + \vec{\nabla} \cdot \vec{A}$

$\pi_i = \partial_i A_0 - \partial_0 A_i = -\dot{A}_i + \vec{\nabla} A^0$

$\Rightarrow [\dot{A}_\rho, \dot{A}_\nu] = 0$

$[\dot{A}_\rho(x), A_\nu(y)]_{ET} = i g_{\rho\nu}\, \delta^{(3)}(\vec{x} - \vec{y})$

$\approx$ 4 SCALAR FIELDS

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2k_0}} \sum_\lambda \left[a^\lambda(k)\,\varepsilon_\mu^\lambda e^{-ik.x} + a^{\lambda+}\varepsilon_\mu^{\lambda*} e^{ik.x}\right]$$

$\Box A^\mu = 0 \Rightarrow$ 4 POLARISATIONS

$$\varepsilon^{(0)} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad \varepsilon^{(3)} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \qquad \vec{k} \parallel z$$

$$\varepsilon^{(1)} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad \varepsilon^{(2)} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

$$\Rightarrow \quad [a^\lambda, a^{\lambda'+}] = -g^{\lambda\lambda'}\,\delta^{(3)}(\vec{k}-\vec{k}')\,(2\pi)^3$$

SHOW THIS

PROBLEM :

$$|1\rangle = a^{0+}|0\rangle$$

$$\langle 1|1\rangle = \langle 0|a^0 a^{0+}|0\rangle$$

$$= -\langle 0|0\rangle$$

INDEFINITE METRIC!

$\Rightarrow$ SELECT $|\psi\rangle$ SUCH AS

$$\partial^\mu A_\mu^+ |\psi\rangle = 0$$

$$|\psi\rangle = |\psi_T\rangle\,|\phi\rangle \qquad \text{AS } i\,\partial.A^+ \sim \sum_{\lambda=0,3} a^\lambda \varepsilon^\lambda.k$$

transverse

time + longitudinal

$\Rightarrow$ WE MUST HAVE

$$[a^0 - a^3]\,|\phi\rangle = 0$$

$$a_3^+|0\rangle = |\phi\rangle$$

$$\langle\phi|\phi\rangle = \langle 0|a_3\,\underline{a_3^+|0\rangle}$$

$$= \langle 0|a_0\,a_3^+|0\rangle = 0$$

$\Rightarrow$ STATES OF ZERO NORM

THEY DO NOT ENTER OBSERVABLES:

$$H = \int d^3x\; \pi^\mu \dot{A}_\mu - \mathcal{L}$$

$$= \frac{1}{2}\int d^3x \sum_1^3 \dot{A}_i^2 + (\vec{\nabla}\cdot\vec{A})^2 - \dot{A}_0^2 - (\nabla A_0)^2$$

$$= \int \frac{d^3k}{(2\pi)^3}\, E_k \left(\sum_{\lambda=1}^3 (a^{\lambda+}a^\lambda) - a^{0+}a^0\right)$$

SHOW THIS

$\langle\psi|H|\psi\rangle$ INDEPENDENT OF a_0, a_3

⚠ TRUE FOR ON-SHELL PHOTONS

OFF-SHELL: $\partial.A \rightarrow k.\varepsilon$

$$g_{\mu\nu} = \sum_\lambda \frac{\varepsilon_\mu^{(\lambda)}\varepsilon_\nu^{(\lambda)}}{\varepsilon^{(\lambda)}.\varepsilon^{(\lambda)}}$$

$\rightarrow \varepsilon_3$ "SURVIVES"

SCALAR BOSONS	FERMIONS	GAUGE FIELDS
$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi \phi^*$	$\bar{\psi}(i\partial\cdot\gamma - m)\psi$	$-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{\lambda}{2}(\partial.A)^2$
$j_\mu = i(\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*)$	$\bar{\psi}\gamma_\mu \psi$	$\partial_\mu F^{\mu\nu}$ broken
$\pi = \dot{\phi}^* \qquad \pi^* = \dot{\phi}$	$\pi = i\psi^\dagger \qquad \pi^\dagger = -i\psi$	$\pi_0 = \partial.A$ $\pi_i = \partial_i A_0 - \partial_0 A_i$
$\phi \sim a e^{-ip.x} + b^\dagger e^{ip.x}$	$\psi \sim a e^{-ip.x} + b^\dagger e^{ip.x}$	
$[a, a^\dagger] = (2\pi)^3 \delta^{(3)} = [b, b^\dagger]$	$\{a, a^\dagger\} = \{b, b^\dagger\} = (2\pi)^3 \delta^{(3)} \delta_{rs}$	$\left\{ \begin{array}{l} a_0^\dagger, a_3^\dagger \mid \psi \rangle \to 0 \\ a_T^\dagger \end{array} \right.$

HOW DO THESE INTERACT

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$$

- CAUSALITY
- LORENTZ INVARIANT
- RENORMALISABLE
- EXTRA SYMETRIES:
  GAUGE, ETC.