6. QED CROSS SECTIONS

- $e^-\mu^- \to e^-\mu^-$
  - spin average and sum
  - $|\overline{\mathcal{M}}|^2$ and crossing
  - cross sections
- $e^+e^- \to e^+e^-$
- $e^-\gamma \to e^-\gamma$

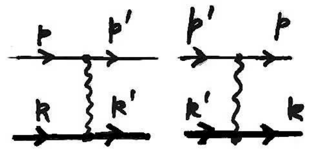
SPIN SUM

$$-i\mathcal{M} = -e^2 \left[\bar{u}(p')\gamma^\mu u(p)\right] \frac{-ig^{\mu\nu}}{q^2+i\varepsilon} \left[\bar{u}(k')\gamma^\nu u(k)\right]$$

$$|\mathcal{M}|^2 = \left[\frac{e^2}{q^2+i\varepsilon}\right]^2 \left[\bar{u}(p')\gamma^\mu u(p)\right]\left[u^\dagger(p)\gamma^{\sigma\dagger}\gamma_0^\dagger u(p')\right] \cdots$$

$$\gamma_0^\dagger = \gamma_0 \qquad \gamma_0\gamma_\mu\gamma_0 = \gamma_\mu^\dagger$$ CHECK THIS

$$= \left[\frac{e^2}{q^2+i\varepsilon}\right] \left[\bar{u}(p')\gamma^\mu u(p)\,\bar{u}(p)\gamma^\sigma u(p')\right]\left[\bar{u}(k')\gamma_\mu u(k)\,\bar{u}(k)\gamma_\sigma u(k')\right]$$



$$\left.\begin{aligned}\sum_s u_s^\alpha(p)\,\bar{u}_s^\beta(p) &= \gamma\cdot p + m \\ \text{Similarly}\quad \sum_\lambda \varepsilon_\lambda^\mu(p)\,\varepsilon_\lambda^{\nu*}(p) &= g^{\mu\nu}\end{aligned}\right\}$$ $1/i$ × top of propagators!

UNPOLARISED BEAM + TARGET
NO SPIN ANALYSIS $\Big\}\ \frac{1}{4}\sum_{\{s_i\}}\sum_{\{s_f\}}$

$$\sum_{s_1 s_2} \bar{u}_\alpha(p',s_1)\,\gamma^\mu_{\alpha\beta}\,\underbrace{u_\beta(p,s_2)\,\bar{u}_\gamma(p,s_2)}_{(\gamma\cdot p+m)_{\beta\gamma}}\,\gamma^\sigma_{\gamma\delta}\,u_\delta(p',s_1)$$

$(\gamma\cdot p'+m)_{\delta\alpha}$

$$= (\gamma\cdot p'+m)_{\delta\alpha}\,\gamma^\mu_{\alpha\beta}\,(\gamma\cdot p+m)_{\beta\gamma}\,\gamma^\sigma_{\gamma\delta} = \mathrm{Tr}\left((\gamma\cdot p'+m)\gamma^\mu(\gamma\cdot p+m)\gamma^\sigma\right)$$
$$\equiv L^{\mu\sigma}$$

$\Rightarrow$ TO CALCULATE σ, WE MUST TAKE TRACES OF γ MATRICES

$$\mathrm{Tr}\,\gamma^\mu = 0$$

$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu = \tfrac{1}{2}\mathrm{Tr}\left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right) = g^{\mu\nu}\,\mathrm{Tr}(1) = 4g^{\mu\nu}$$

$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\sigma = \mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma_5^2\right) = \mathrm{Tr}\left(\gamma_5\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^5\right) = -\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\sigma\right) = 0$$

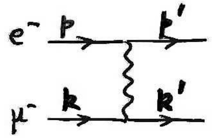
$$\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) = \left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)4$$

show this

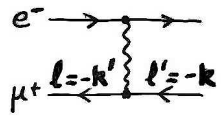
$$L^{\mu\sigma} = \mathrm{Tr}\left((\gamma\cdot p' + m)\gamma^\mu(\gamma\cdot p + m)\gamma^\sigma\right)$$
$$= \mathrm{Tr}\left(\gamma\cdot p'\gamma^\mu\gamma\cdot p\gamma^\sigma\right) + m^2\,\mathrm{Tr}\left(\gamma^\mu\gamma^\sigma\right)$$
$$= \left(p'^\mu p^\sigma + p'^\sigma p^\mu - p\cdot p' g^{\mu\sigma} + m^2 g^{\mu\sigma}\right)\times 4$$

$$|\mathcal{M}|^2 = \left(\frac{e^2}{q^2}\right)^2 L^{\mu\sigma}_{(e)}\,L^{(\mu)}_{\mu\sigma}$$
$$= \left(\frac{e^2}{q^2}\right)^2\left(p'^\mu p^\sigma + p'^\sigma p^\mu + (m_e^2 - p\cdot p')g^{\mu\sigma}\right)\left(k'_\mu k_\sigma + k'_\sigma k_\mu + (m_\mu^2 - k\cdot k')g_{\mu\sigma}\right)$$

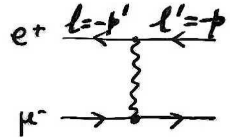
$$= \left(\frac{2e^2}{q^2}\right)^2\left(p'\cdot k'\,p\cdot k + p'\cdot k\,p\cdot k' + \cancel{(m_e^2 - p\cdot p')\,k\cdot k'} + \cancel{(m_\mu^2 - k\cdot k')\,p\cdot p'} + 2(m_e^2 - \cancel{p\cdot p'})(m_\mu^2 - \cancel{k\cdot k'})\right)$$
$$= \left(\frac{2e^2}{q^2}\right)^2\left(p'\cdot k'\,p\cdot k + p'\cdot k\,p\cdot k' + 2m_e^2 m_\mu^2\right)$$



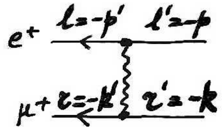
$$\frac{1}{(p-p')^2}\left(p'\cdot k'\,p\cdot k + p'\cdot k\,p\cdot k' + 2m_e^2 m_\mu^2\right)$$



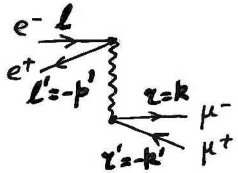
$$\frac{1}{(p-p')^2}\left(p'\cdot l\,p\cdot l' + p'\cdot l'\,p\cdot l + 2m_e^2 m_\mu^2\right) \quad \text{same}$$



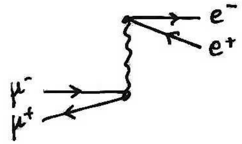
$$\frac{1}{(l'-l)^2}\left(l\cdot k'\,l'\cdot k + l'\cdot k'\,l\cdot k + 2m_e^2 m_\mu^2\right) \quad \text{same}$$



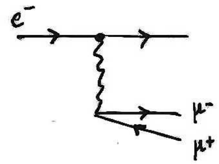
$$\frac{1}{(l'-l)^2}\left(l\cdot r\,l'\cdot r' + l\cdot r'\,l'\cdot r + 2m_e^2 m_\mu^2\right) \quad \text{same}$$



$$\frac{1}{(l+l')^2}\left(+l'\cdot r'\,l\cdot r + l\cdot r'\,l'\cdot r + 2m_e^2 m_\mu^2\right) \neq$$



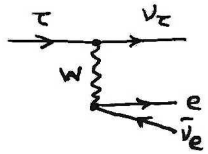
same as above $\neq$



$$\frac{1}{(p-p')^2}\left(-p'\cdot k'\,p\cdot k - p\cdot k'\,p'\cdot k + 2m_e^2 m_\mu^2\right)$$

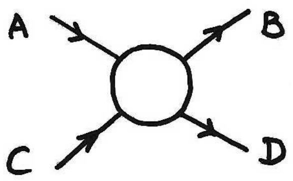
forbidden by $\delta^{(4)}(p - p' - k - k')$

BUT



exists

MANDELSTAM INVARIANTS

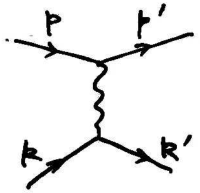


$$s = (p_A + p_C)^2 = (p_B + p_D)^2 = E_{cm}^2$$

$$t = (p_A - p_B)^2 = (p_C - p_D)^2$$

$$u = (p_A - p_D)^2 = (p_C - p_B)^2$$

$$s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2$$ SHOW THIS



$$s = (p+k)^2 = (p'+k')^2$$
$$t = (p-p')^2$$
$$u = (p-k')^2$$

$$|\overline{\mathcal{M}}|^2 = \frac{2e^4}{t^2}\left(\left(\frac{s}{2}\right)^2 + \left(\frac{u}{2}\right)^2\right) .$$ $m_e, m_\mu \ll \sqrt{s}$

c.m.: $t = 2p\cdot p' = 2\frac{\sqrt{s}}{2}(1-\cos\theta)$

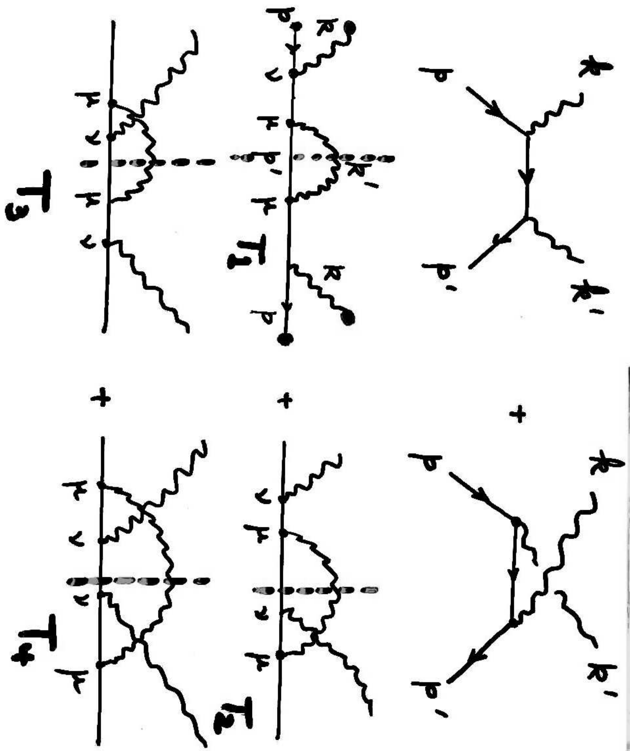
$u = 2p\cdot k' = \sqrt{s}(1+\cos\theta)$

$$\Rightarrow \quad \frac{d\sigma}{d\cos\theta\, d\varphi} = \frac{|\overline{\mathcal{M}}|^2}{64\pi^2 s} \Rightarrow \frac{d\sigma}{d\cos\theta} = \frac{2e^4}{32\pi s}\frac{1}{4}\frac{1+(1+\cos\theta)^2}{(1-\cos\theta)^2}$$

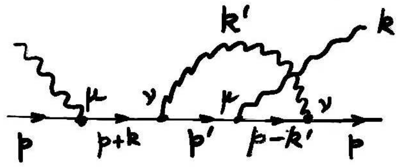
$$\boxed{\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{4}\frac{2+2\cos\theta+\cos^2\theta}{(1-\cos\theta)^2}}$$

singular at $t=0$

Compton scattering



T_2 :



$$\frac{\mathrm{Tr}\left(p\cdot\gamma\,\gamma^\nu (p-k')\cdot\gamma\,\gamma^\mu\, p'\cdot\gamma\,\gamma_\nu\,(p+k)\cdot\gamma\,\gamma_\mu\right)}{(p+k)^2(p-k')^2}$$

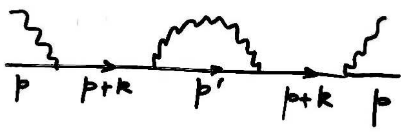
$$= \frac{-2\,\mathrm{Tr}\left(p\cdot\gamma\,\gamma^\nu (p-k')\cdot\gamma\,(p+k)\cdot\gamma\,\gamma_\nu\, p'\cdot\gamma\right)}{su}$$

$$= \frac{8\,\mathrm{Tr}\left(p\cdot\gamma\; p'\cdot\gamma\right)(p-k')\cdot(p+k)}{su} = \frac{32\,(p\cdot p')\left(\underset{-t/2}{-k'\cdot p}+\underset{+u/2}{k\cdot p}-\underset{s/2}{k'\cdot k}\right)}{su} \quad \tfrac{t}{2}$$

$$= 0 \qquad (\sim m^2)$$

similarly $T_3 \simeq 0$

T_1



$$= \frac{\mathrm{Tr}\left(p\cdot\gamma\,\gamma^\nu (p+k)\cdot\gamma\,\gamma^\mu (p'\cdot\gamma)\,\gamma_\mu\,(p+k)\cdot\gamma\,\gamma_\nu\right)}{((p+k)^2)^2} \times (-1)\times e^4$$

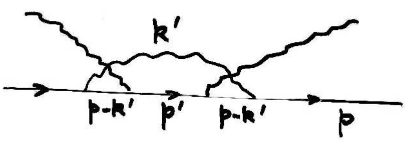
$$= \frac{(-2)^2\,\mathrm{Tr}\left(p\cdot\gamma\,(p+k)\cdot\gamma\; p'\cdot\gamma\,(p+k)\cdot\gamma\right)}{((p+k)^2)^2}\,(-e^4)$$

$$= 4\times 4\,\frac{p\cdot(\cancel{p}+k)\,p'\cdot(p+k) - p\cdot p'\,(p+k)^2 + p\cdot(\cancel{p}+k)\,p'\cdot(p+k)}{((p+k)^2)^2}(-e^4) \quad s/2$$

$$= 16\,\frac{2\,p\cdot k\,\cancel{p'\cdot p} + 2\,p\cdot k\,p'\cdot k - \cancel{2p\cdot p'\,p\cdot k}}{((p+k)^2)^2}(-e^4) = 32\,\frac{\underset{s/2}{p\cdot k}\;\underset{u/2}{p'\cdot k}}{s^2}(-e^4)$$

$$= 8\,\frac{u}{s}(-e^4)$$

T_4



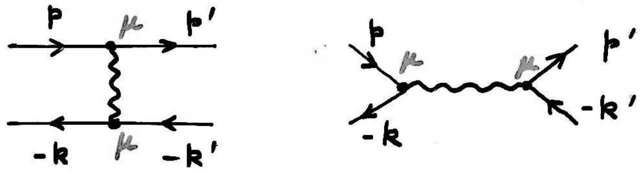
same as T_1 if $p+k$ is replaced by $p-k'$

$k \leftrightarrow -k' \qquad s \leftrightarrow u$

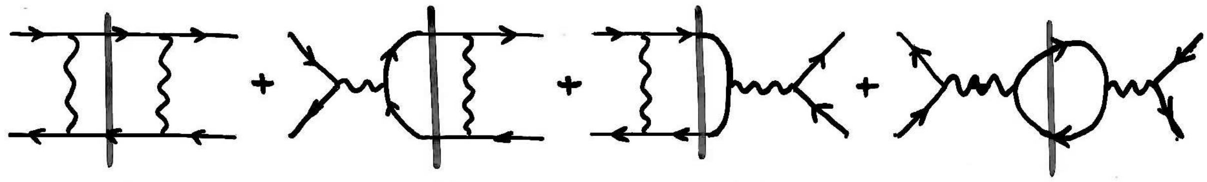
$$T_4 = 32(-e^4)\,\frac{p\cdot k'\; p'\cdot k'}{(p-k')^2} = 8\,\frac{s}{u}(-e^4)$$

$e^+e^- \longrightarrow e^+e^-$ (Bhabha)

amplitudes $i\mathcal{M}$



squares $|\mathcal{M}|^2$ summed on spins



$$s = (p+k)^2 = (p'+k')^2 = 2p.k = 2p'.k' \qquad (m_e \simeq 0)$$

$$t = (p-p')^2 = (k-k')^2 = -2p.p' = -2k.k'$$

$$u = (p-k')^2 = (p'-k)^2 = -2p.k' = -2p'.k$$

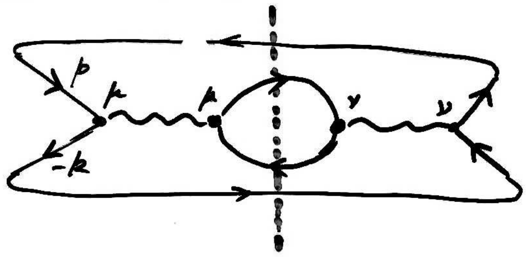
$$\Rightarrow \quad \overline{|\mathcal{M}|^2_c} = (-e^4)\,\frac{8}{4}\left(\frac{u}{s} + \frac{s}{u}\right)$$

$$\frac{d\sigma}{d\cos\theta} = -\frac{1}{32\pi s}\,2e^4\left(\frac{u}{s} + \frac{s}{u}\right)$$

$$= \frac{-1}{32\pi s}\,2e^4\left(\frac{-\frac{s}{2}(1+\cos\theta)}{s} + \frac{s}{-\frac{s}{2}(1+\cos\theta)}\right)$$

$$= \frac{e^4}{16\pi s}\left(\frac{1+\cos\theta}{2} + \frac{2}{1+\cos\theta}\right)$$

T_4



2 loops $(-1)^2$

$$T_4 = \frac{Tr\left(\gamma^\mu\, p\cdot\gamma\, \gamma^\nu (-k\cdot\gamma)\right)\ Tr\left(\gamma_\mu (-k'\cdot\gamma)\,\gamma_\nu (-p'\cdot\gamma)\right)}{s^2}$$

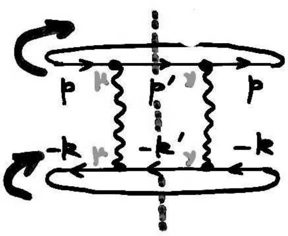
$$= \frac{Num\left(T_1\left(p' \leftrightarrow -k\right)\right)}{s^2}$$

$p' \leftrightarrow -k$

$s \leftrightarrow t \qquad u \to u$

$$= \frac{8\left(t^2 + u^2\right)}{s^2}$$

T_1 :



2 loops $\Rightarrow (-1)^2$

$$|\mathcal{M}_1|^2 = \frac{Tr\left(ie\gamma^\mu\, p\cdot\gamma\ ie\gamma^\nu\, p'\cdot\gamma\right)\ Tr\left(ie\gamma_\mu (-k'\cdot\gamma)\, ie\gamma_\nu (-k\cdot\gamma)\right)}{(p-p')^4}$$

$$= e^4\,\frac{16\left(p^\mu p'^\nu + p^\nu p'^\mu - g^{\mu\nu}\, p\cdot p'\right)\left(k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}\, k\cdot k'\right)}{t^2}$$

$$= \frac{16e^4}{t^2}\Big(\underline{p.k\,p'.k'} + p.k'\,p'.k - \underline{p.p'\,k.k'} + p.k'\,p'.k + \underline{p.k\,p'.k'} - \underline{p.p'\,k.k'} - \underline{k.k'\,p.p'} - k.k'\,p.p' + \underline{4\,p.p'\,k.k'}\Big)$$

$$= \frac{32e^4}{t^2}\left(\left(\frac{s}{2}\right)^2 + \left(\frac{u}{2}\right)^2\right) \quad = 8e^4\ \frac{s^2+u^2}{t^2}$$

Finally:

$$\overline{|\mathcal{M}|}^2_{Bh.} = \frac{|\mathcal{M}|^2}{4} = 2e^4\left(\frac{s^2+u^2}{t^2} + \frac{2u^2}{ts} + \frac{u^2+t^2}{s^2}\right)$$

$$= 2e^4\left(\left(\frac{s}{t}\right)^2 + \left(\frac{t}{s}\right)^2 + \left(\frac{u}{s}+\frac{u}{t}\right)^2\right)$$

$$\frac{d\sigma}{d\Omega} = \frac{\overline{|\mathcal{M}|}^2}{64\pi^2 s}$$

$$p = \left(\frac{\sqrt{s}}{2}, \vec{0}, \frac{\sqrt{s}}{2}\right) \qquad k = \left(\frac{\sqrt{s}}{2}, \vec{0}, -\frac{\sqrt{s}}{2}\right)$$

$$p' = \left(\frac{\sqrt{s}}{2}, \dots, \frac{\sqrt{s}}{2}\cos\theta\right) \qquad k' = \left(\frac{\sqrt{s}}{2}, \dots, -\frac{\sqrt{s}}{2}\cos\theta\right)$$

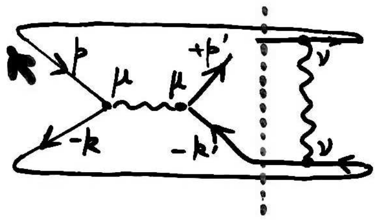
$$t = (p-p')^2 = -2p.p' = -2\frac{s}{4}(1-\cos\theta)$$

$$u = (p-k')^2 = -\frac{s}{2}(1+\cos\theta)$$

$$\Rightarrow \frac{d\sigma}{d\cos\theta} = \frac{2e^4}{32\pi s}\left\{\frac{4}{(1-\cos\theta)^2} + \frac{(1-\cos\theta)^2}{4} + \left(\frac{(1+\cos\theta)}{2} + \frac{1+\cos\theta}{1-\cos\theta}\right)^2\right\}$$

cf. Rutherford as $\theta \to 0$.

T_2



1 loop $\Rightarrow$ -1

$$T_2 = e^4 \frac{Tr(\gamma^\mu p.\gamma\, \gamma^\nu p'.\gamma\, \gamma_\mu (-k'.\gamma)\, \gamma_\nu (-k.\gamma))}{st} \times (-1)$$

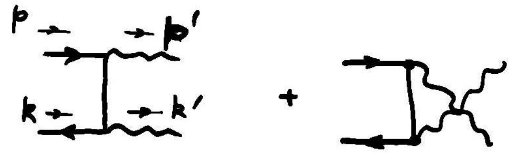
$$= e^4 \frac{Tr(\gamma^\mu p.\gamma\, (-2\, k'.\gamma\, \gamma_\mu\, p'.\gamma)(+k.\gamma))}{st} \times (-1)$$

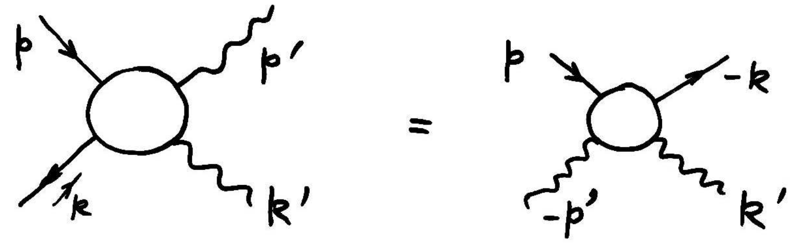
$$= 8e^4 \frac{Tr(k'.p\;\, p'.\gamma\, k.\gamma)}{st}$$

$$= 32e^4 \frac{k'.p\;\, k.p'}{st} = \frac{8e^4 u^2}{st}$$

$$T_3 = T_2(p \leftrightarrow -k',\; p' \leftrightarrow -k) = T_2$$

$e^+e^- \to \gamma\gamma$





$= \text{Compton } (k \to -p')$

crois. fermion

$s = (p+k)^2 \leftrightarrow (p-p')^2 = t$

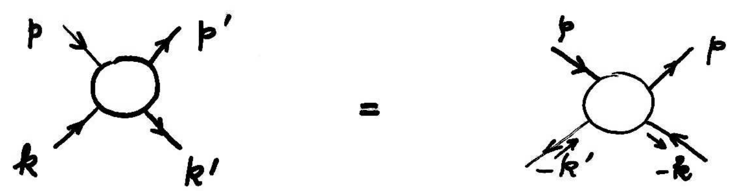
$$\Rightarrow |\overline{\mathcal{M}}|^2 = (+e)^4 \, 2\left(\frac{u}{t} + \frac{t}{u}\right)\frac{1}{4}$$

$$= +2e^4 \left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right)\frac{1}{4}$$

$$= +2e^4 \frac{1}{4}\left(\frac{2+2\cos^2\theta}{\sin^2\theta}\right) = e^4\left(\frac{1+\cos^2\theta}{\sin^2\theta}\right)$$

$$\frac{d\sigma}{d\theta} = \frac{e^4}{32\pi s}\frac{1+\cos^2\theta}{\sin^2\theta}$$

Møller scattering $\qquad e^-e^- \to e^-e^-$



$s = (p+k)^2 \longleftrightarrow u = (p-k')^2$

$$|\mathcal{M}|^2_{Moller}(s,t,u) = |\mathcal{M}|^2_{Bh}(u,t,s)$$

$$= 2\left[\frac{u^2}{t^2} + \frac{t^2}{u^2} + \left(\frac{s}{u} + \frac{s}{t}\right)^2\right]$$