6. QED CROSS SECTIONS

- $e^-\mu^- \to e^-\mu^-$
  - spin average and sum
  - $|\overline{\mathcal{M}}|^2$ and crossing
  - cross sections
- $e^+e^- \to e^+e^-$
- $e^-\gamma \to e^-\gamma$

SPIN SUM

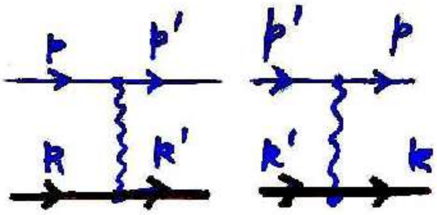
$$-i\mathcal{M} = -e^2 \left[\bar{u}(p')\gamma^\mu u(p)\right]\frac{-ig^{\mu\nu}}{q^2+i\epsilon}\left[\bar{u}(k')\gamma^\nu u(k)\right]$$

$$|\mathcal{M}|^2 = \left[\frac{e^2}{q^2+i\epsilon}\right]^2 \left[\bar{u}(p')\gamma^\mu u(p)\right]\left[u^\dagger(p)\gamma^{\sigma\dagger}\gamma_0^\dagger u(p')\right]\cdots$$

$$\gamma_0^\dagger = \gamma_0 \qquad \gamma_0\gamma_\mu\gamma_0 = \gamma_\mu^\dagger$$

CHECK THIS

$$= \left[\frac{e^2}{q^2+i\epsilon}\right]\left[\bar{u}(p')\gamma^\mu u(p)\,\bar{u}(p)\gamma^\sigma u(p')\right]\left[\bar{u}(k')\gamma_\mu u(k)\,\bar{u}(k)\gamma_\sigma u(k')\right]$$



$$\sum_s u_s^\alpha(p)\,\bar{u}_s^\beta(p) = \gamma\cdot p + m$$

Similarly $\sum_\lambda \varepsilon_\lambda^\mu(p)\,\varepsilon_\lambda^{\nu *}(p) = g^{\mu\nu}$

$\Big\}$ $1/i$ × top of propagators!

UNPOLARISED BEAM + TARGET
NO SPIN ANALYSIS $\Big\}$ $\frac{1}{4}\sum_{\{s_i\}}\sum_{\{s_f\}}$

$$\sum_{s_1 s_2} \bar{u}_\alpha(p', s_1)\,\gamma^\mu_{\alpha\beta}\,\underbrace{u_\beta(p, s_2)\,\bar{u}_\gamma(p, s_2)}_{(\gamma\cdot p+m)_{\beta\gamma}}\,\gamma^\sigma_{\gamma\delta}\,u_\delta(p', s_1)$$

$$(\gamma\cdot p'+m)_{\delta\alpha}$$

$$= (\gamma\cdot p'+m)_{\delta\alpha}\,\gamma^\mu_{\alpha\beta}\,(\gamma\cdot p+m)_{\beta\gamma}\,\gamma^\sigma_{\gamma\delta} = \mathrm{Tr}\left((\gamma\cdot p'+m)\gamma^\mu(\gamma\cdot p+m)\gamma^\sigma\right)$$

$$\equiv L^{\mu\sigma}$$

$\Rightarrow$ TO CALCULATE σ, WE MUST TAKE TRACES OF γ MATRICES

$$\mathrm{Tr}\,\gamma^\mu = 0$$

$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu = \tfrac{1}{2}\mathrm{Tr}\left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right) = g^{\mu\nu}\,\mathrm{Tr}(1)$$
$$= 4\,g^{\mu\nu}$$

$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\sigma = \mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma_5^2\right) = \mathrm{Tr}\left(\gamma_5\gamma^\mu\gamma^\nu\gamma^\sigma\gamma_5\right)$$
$$= -\,\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\sigma\right) = 0$$

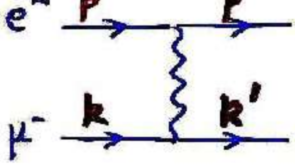
$$\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) = \left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)4$$

show this

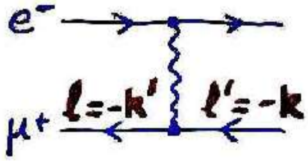
$$L^{\mu\sigma} = \mathrm{Tr}\left((\gamma\cdot p' + m)\,\gamma^\mu\,(\gamma\cdot p + m)\,\gamma^\sigma\right)$$
$$= \mathrm{Tr}\left(\gamma\cdot p'\,\gamma^\mu\,\gamma\cdot p\,\gamma^\sigma\right) + m^2\,\mathrm{Tr}\left(\gamma^\mu\gamma^\sigma\right)$$
$$= \left(p'^\mu p^\sigma + p'^\sigma p^\mu - p\cdot p'\,g^{\mu\sigma} + m^2 g^{\mu\sigma}\right)\times 4$$

$$|\bar{\mathcal{M}}|^2 = \left(\frac{e^2}{q^2}\right)^2 L^{\mu\sigma}_{(e)}\,L^{(\mu)}_{\mu\sigma}$$
$$= \left(\frac{e^2}{q^2}\right)^2\left(p'^\mu p^\sigma + p'^\sigma p^\mu + (m_e^2 - p\cdot p')\,g^{\mu\sigma}\right)\left(k'_\mu k_\sigma + k'_\sigma k_\mu + (m_\mu^2 - k\cdot k')\,g_{\mu\sigma}\right)$$

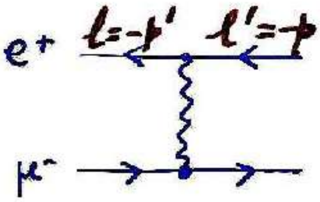
$$= 2\left(\frac{e^2}{q^2}\right)^2\left(p'\cdot k'\,p\cdot k + p'\cdot k\,p\cdot k' + \cancel{(m_e^2 - p\cdot p')\,k\cdot k'} + \cancel{(m_\mu^2 - k\cdot k')\,p\cdot p'} + 2\,(m_e^2 - \cancel{p\cdot p'})(m_\mu^2 - \cancel{k\cdot k'})\right)$$
$$= 2\left(\frac{e^2}{q^2}\right)^2\left(p'\cdot k'\,p\cdot k + p'\cdot k\,p\cdot k' + 2\,m_e^2 m_\mu^2\right)$$



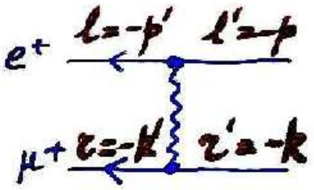
$$\frac{1}{(p-p')^2}\left(p'.k'\,p.k + p'.k\,p.k' + 2m_e^2 m_\mu^2\right)$$



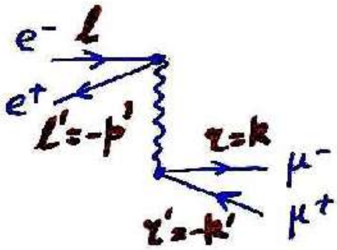
$$\frac{1}{(p-p')^2}\left(p'.l\,p.l' + p'.l'\,p.l + 2m_e^2 m_\mu^2\right) \quad \text{same}$$



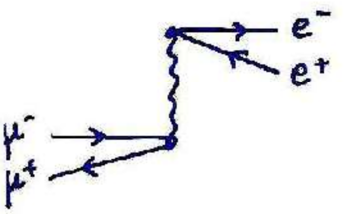
$$\frac{1}{(l'-l)^2}\left(l.k'\,l'.k + l'.k'\,l.k + 2m_e^2 m_\mu^2\right) \quad \text{same}$$



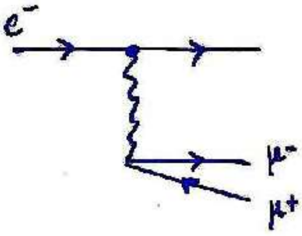
$$\frac{1}{(l'-l)^2}\left(l.\tau\,l'.\tau' + l.\tau'\,l'.\tau + 2m_e^2 m_\mu^2\right) \quad \text{same}$$



$$\frac{1}{(l+l')^2}\left(+l'.\tau'\,l.\tau + l.\tau'\,l'.\tau + 2m_e^2 m_\mu^2\right) \neq$$

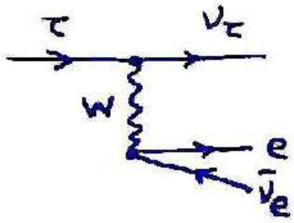


same as above $\neq$



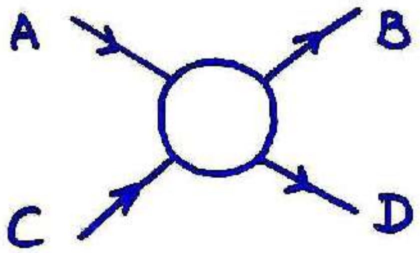
$$\frac{1}{(p-p')^2}\left(-p'.k'\,p.k - p.k'\,p'.k + 2m_e^2 m_\mu^2\right)$$

forbidden by $\delta^{(4)}(p-p'-k-k')$

BUT 

exists

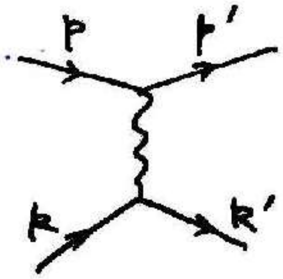
MANDELSTAM INVARIANTS



$$s = (p_A + p_C)^2 = (p_B + p_D)^2 = E_{cm}^2$$
$$t = (p_A - p_B)^2 = (p_C - p_D)^2$$
$$u = (p_A - p_D)^2 = (p_C - p_B)^2$$

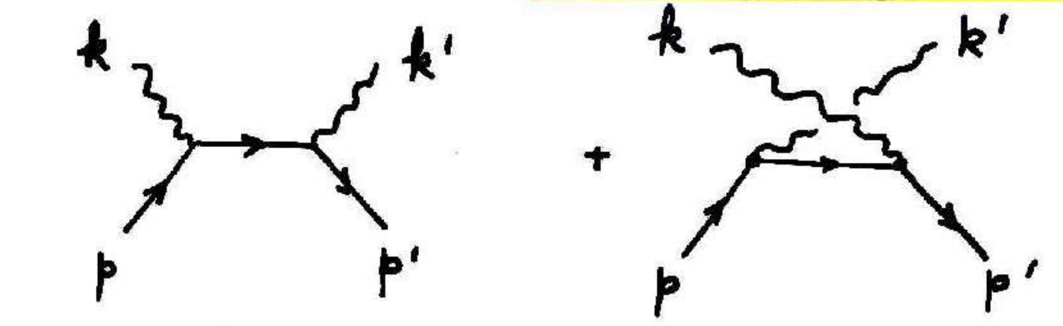
$$s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2$$

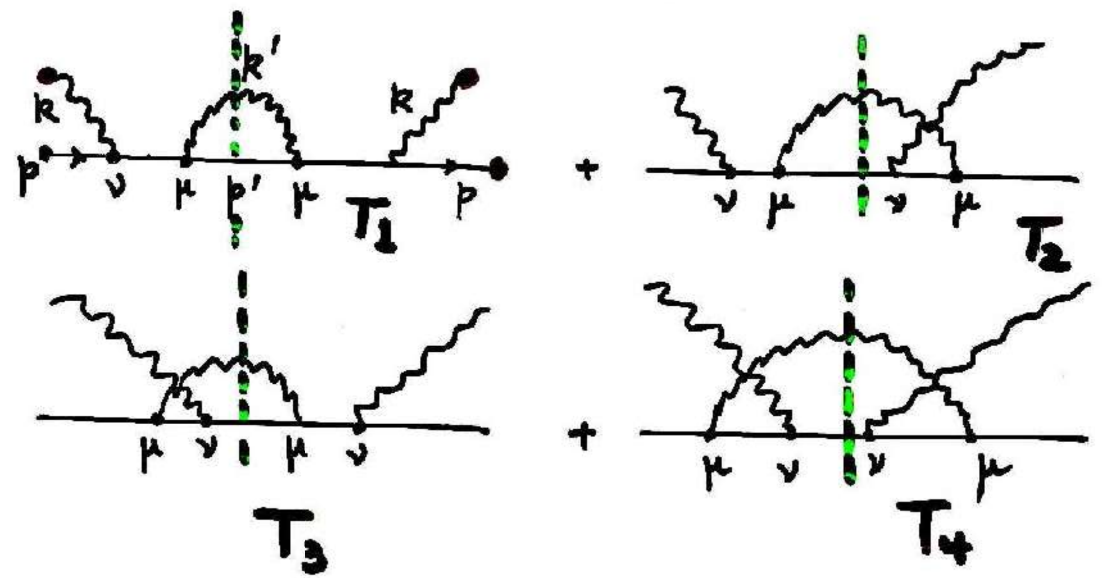
SHOW THIS



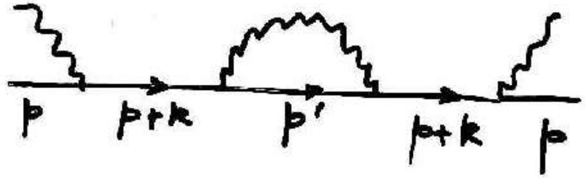
$$s = (p+k)^2 = (p'+k')^2$$
$$t = (p-p')^2$$
$$u = (p-k')^2$$

Compton scattering





T_1



$$= \frac{Tr\left(p\cdot\gamma\ \gamma^{\nu}\,(p+k)\cdot\gamma\ \gamma^{\mu}(p'\cdot\gamma)\,\gamma_{\mu}\,(p+k)\cdot\gamma\ \gamma_{\nu} \right)}{((p+k)^2)^2} \times (-1)\times e^4$$

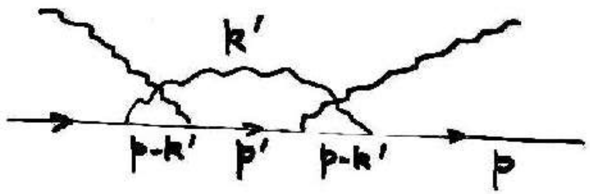
$$= \frac{(-2)^2 Tr\ \left(p\cdot\gamma\ (p+k)\cdot\gamma\ p'\cdot\gamma\ (p+k)\cdot\gamma \right)}{((p+k)^2)^2}\ (-e^4)$$

$$= 4\times 4\ \frac{p\cdot(\cancel{p}+k)\,p'\cdot(p+k) - p\cdot p'\,(p+k)^2 + p\cdot(\cancel{p}+k)\,p'\cdot(p+k)}{((p+k)^2)^2}(-e^4)$$

$$= 16\ \frac{2\,p\cdot k\,\cancel{p'\cdot p} + 2\,p\cdot k\,p'\cdot k - 2\cancel{p\cdot p'}\,p\cdot k}{((p+k)^2)^2}(-e^4) = 32\ \frac{\overset{s/2}{p\cdot k}\ \overset{u/2}{p'\cdot k}}{s^2}\,(-e^4)$$

$$= 8\ \frac{u}{s}\,(-e^4)$$

T_4

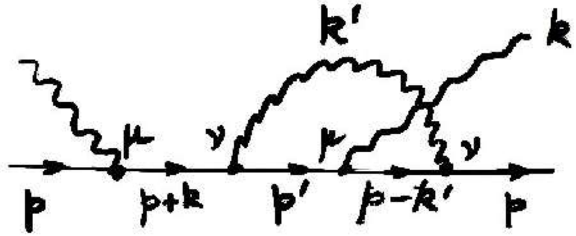


same as T_1 if $p+k$ is replaced by $p-k'$

or $k \to -k'$ $\quad s \to u$

$$T_4 = 32(-e^4)\,\frac{p\cdot k'\ p'\cdot k'}{(p-k')^2} = 8\ \frac{s}{u}\,(-e^4)$$

T_2 :



$$\frac{\mathrm{Tr}\left(p\cdot\gamma\,\gamma^{\nu}\,(p-k')\cdot\gamma\,\overbrace{\gamma^{\mu}\,p'\cdot\gamma\,\gamma_{\nu}\,(p+k)\cdot\gamma\,\gamma_{\mu}}\right)}{(p+k)^2\,(p-k')^2}$$

$$= \frac{-2\,\mathrm{Tr}\left(p\cdot\gamma\,\overbrace{\gamma^{\nu}\,(p-k')\cdot\gamma\,(p+k)\cdot\gamma\,\gamma_{\nu}}\,p'\cdot\gamma\right)}{su}$$

$$= \frac{8\,\mathrm{Tr}\left(p\cdot\gamma\,p'\cdot\gamma\right)(p-k')\cdot(p+k)}{su} = \frac{32\,(p\cdot p')\left(\overset{-t/2}{-k'\cdot p}+\overset{+u/2}{k\cdot p}\overset{s/2}{-}\overset{t/2}{k'\cdot k}\right)}{su}$$

$$= 0 \qquad (\sim m^2)$$

similarly $\quad T_3 \simeq 0$

$$\Rightarrow \quad \left|\overline{\mathcal{M}}\right|_c^2 = (-e^4)\,\frac{8}{4}\left(\frac{u}{s}+\frac{s}{u}\right)$$

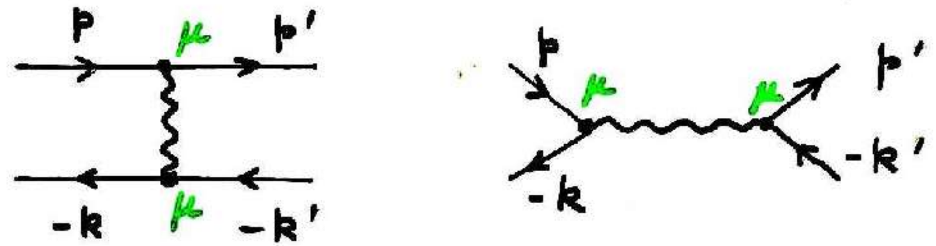
$$\frac{d\sigma}{d\cos\theta} = -\frac{1}{32\pi s}\;2e^4\left(\frac{u}{s}+\frac{s}{u}\right)$$

$$= \frac{-1}{32\pi s}\;2e^4\left(\frac{-\frac{s}{2}(1+\cos\theta)}{s}+\frac{s}{-\frac{s}{2}(1+\cos\theta)}\right)$$

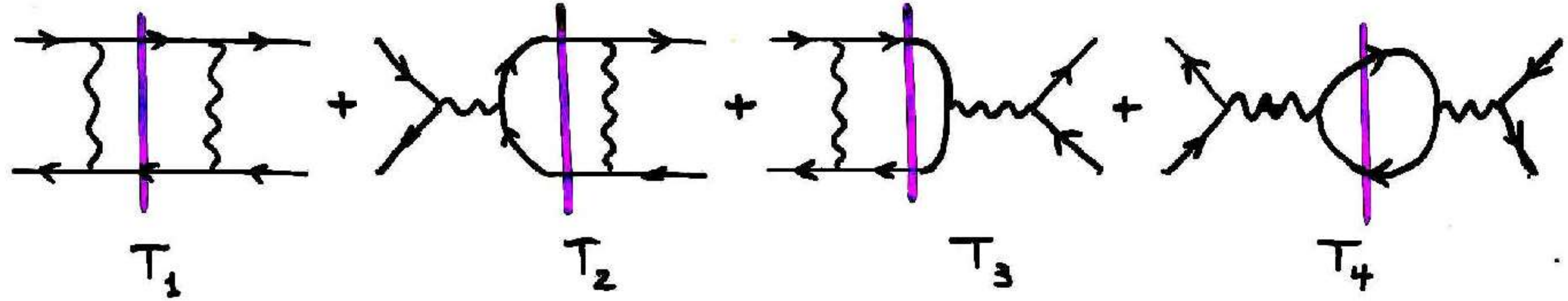
$$= \frac{e^4}{16\pi s}\left(\frac{1+\cos\theta}{2}+\frac{2}{1+\cos\theta}\right)$$

$e^+e^- \rightarrow e^+e^-$ (Bhabha)

amplitudes $i\mathcal{M}$



squares $|\mathcal{M}|^2$ summed on spins

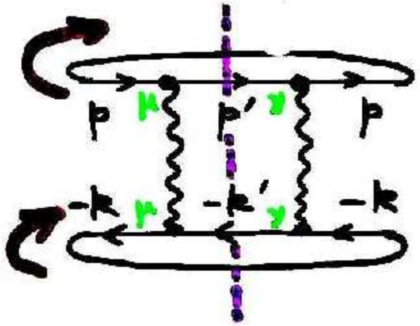


$$s = (p+k)^2 = (p'+k')^2 = 2p.k = 2p'.k' \qquad (m_e \simeq 0)$$

$$t = (p-p')^2 = (k-k')^2 = -2p.p' = -2k.k'$$

$$u = (p-k')^2 = (p'-k)^2 = -2p.k' = -2p'.k$$

T_1 :



2 loops $\Rightarrow (-1)^2$

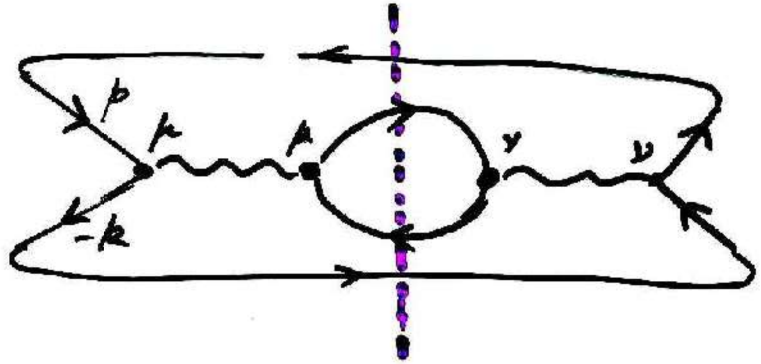
$$|\mathcal{M}_1|^2 = \frac{Tr\left(ie\gamma^\mu\, p.\gamma\;\; ie\gamma^\nu\, p'.\gamma\right)\; Tr\left(ie\gamma_\mu (-k'.\gamma) ie\gamma_\nu\, (-k.\gamma)\right)}{(p-p')^4}$$

$$= e^4 \frac{16\left(p^\mu p'^\nu + p^\nu p'^\mu - g^{\mu\nu} p.p'\right)\left(k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu} k.k'\right)}{t^2}$$

$$= \frac{16e^4}{t^2}\Big(p.k\, p'.k' + p.k'\, p'.k - p.p'\, k.k' + p.k'\, p'.k + p.k\, p'.k' - p.p'\, k.k' - k.k'\, p.p' - k.k'\, p.p'$$
$$+ 4\, p.p'\, k.k'\Big)$$

$$= \frac{32e^4}{t^2}\left(\left(\frac{s}{2}\right)^2 + \left(\frac{u}{2}\right)^2\right) \quad = 8e^4\, \frac{s^2+u^2}{t^2}$$

T_4



2 loops $(-1)^2$

$$T_4 = \frac{Tr\left(\gamma^\mu\, p\cdot\gamma\, \gamma^\nu (-k\cdot\gamma)\right)\; Tr\left(\gamma_\mu (-k'\cdot\gamma)\, \gamma_\nu (-p'\cdot\gamma)\right)}{s^2}$$

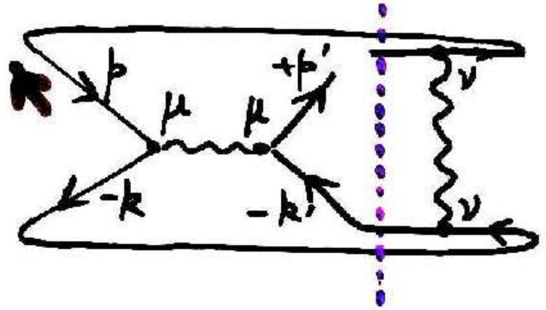
$$= \frac{Num\left(T_1\left(p' \leftrightarrow -k\right)\right)}{s^2}$$

$p' \leftrightarrow -k$

$s \leftrightarrow t \qquad u \to u$

$$= \frac{8\left(t^2 + u^2\right)}{s^2}$$

T_2



1 loop ⇒ −1

$$T_2 = e^4 \frac{\mathrm{Tr}\left(\gamma^\mu\, p\cdot\gamma\, \gamma^\nu\, p'\cdot\gamma\, \gamma_\mu\, (-k'\cdot\gamma)\, \gamma_\nu\, (-k\cdot\gamma)\right)}{st} \times (-1)$$

$$= e^4 \frac{\mathrm{Tr}\left(\gamma^\mu\, p\cdot\gamma\, \left(-2\, k'\cdot\gamma\, \gamma_\mu\, p'\cdot\gamma\right)(+k\cdot\gamma)\right)}{st} \times (-1)$$

$$= 8e^4 \frac{\mathrm{Tr}\left(k'\cdot p\;\, p'\cdot\gamma\; k\cdot\gamma\right)}{st}$$

$$= 32e^4 \frac{k'\cdot p\;\; k\cdot p'}{st} = 8e^4 \frac{u^2}{st}$$

$$T_3 = T_2\left(p \leftrightarrow -k',\; p' \leftrightarrow -k\right) = T_2$$

Finally:

$$\overline{|\mathcal{M}|}^2_{\text{Bh.}} = \frac{|\mathcal{M}|^2}{4} = 2e^4\left(\frac{s^2+u^2}{t^2} + \frac{2u^2}{ts} + \frac{u^2+t^2}{s^2}\right)$$

$$= 2e^4\left(\left(\frac{s}{t}\right)^2 + \left(\frac{t}{s}\right)^2 + \left(\frac{u}{s} + \frac{u}{t}\right)^2\right)$$

$$\frac{d\sigma}{d\Omega} = \frac{\overline{|\mathcal{M}|}^2}{64\pi^2 s}$$

$$p = \left(\frac{\sqrt{s}}{2}, \vec{0}, \frac{\sqrt{s}}{2}\right) \qquad k = \left(\frac{\sqrt{s}}{2}, \vec{0}, -\frac{\sqrt{s}}{2}\right)$$

$$p' = \left(\frac{\sqrt{s}}{2}, \dots, \frac{\sqrt{s}}{2}\cos\theta\right) \qquad k' = \left(\frac{\sqrt{s}}{2}, \dots, -\frac{\sqrt{s}}{2}\cos\theta\right)$$

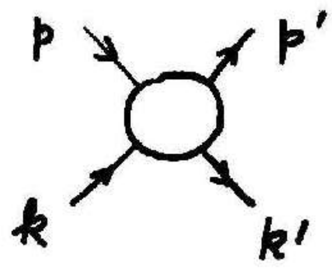
$$t = (p-p')^2 = -2p.p' = -2\frac{s}{4}(1-\cos\theta)$$

$$u = (p-k')^2 = -\frac{s}{2}(1+\cos\theta)$$

$$\Rightarrow \frac{d\sigma}{d\cos\theta} = \frac{2e^4}{32\pi s}\left\{\frac{4}{(1-\cos\theta)^2} + \frac{(1-\cos\theta)^2}{4} + \left(\frac{(1+\cos\theta)}{2} + \frac{1+\cos\theta}{1-\cos\theta}\right)^2\right\}$$

cf Rutherford as $\theta \to 0$

Møller scattering $\qquad e^-e^- \to e^-e^-$



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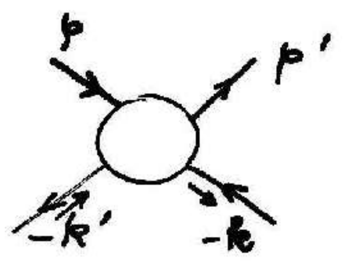
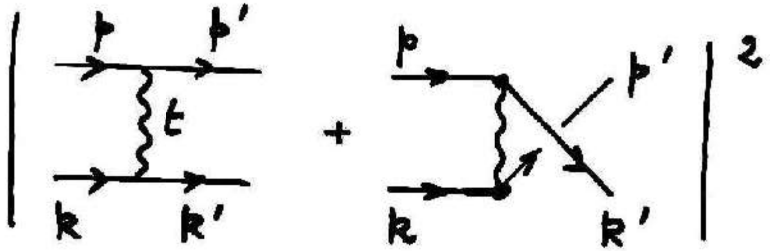


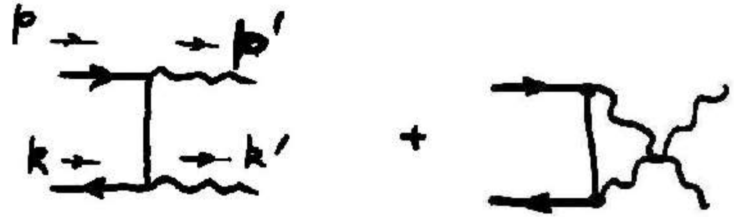
$$s = (p+k)^2 \longleftrightarrow u = (p-k')^2$$

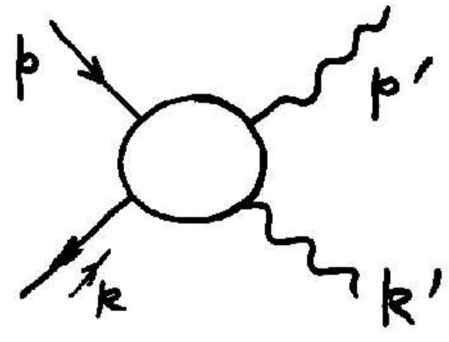
$$|\mathcal{M}|^2_{Moller}(s,t,u) = |\mathcal{M}|^2_{Bh}(u,t,s)$$

$$= 2\left[\frac{u^2}{t^2} + \frac{t^2}{u^2} + \left(\frac{s}{u} + \frac{s}{t}\right)^2\right]$$



$e^+e^- \longrightarrow \gamma\gamma$





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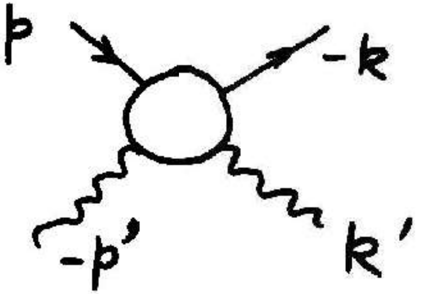

$= \text{Compton } (k \to -p')$

$s = (p+k)^2 \longleftrightarrow (p-p')^2 = t$

Crois. fermion

$$\Rightarrow \quad |\overline{\mathcal{M}}|^2 = (+e)^4 \, 2\left(\frac{u}{t} + \frac{t}{u}\right)\frac{1}{4}$$

$$= +2e^4 \left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right)\frac{1}{4}$$

$$= +2e^4 \frac{1}{4}\left(\frac{2+2\cos^2\theta}{\sin^2\theta}\right) = e^4 \left(\frac{1+\cos^2\theta}{\sin^2\theta}\right)$$

$$\frac{d\sigma}{d\theta} = \frac{e^4}{32\pi s} \; \frac{1+\cos^2\theta}{\sin^2\theta}$$