5. INTERACTING FIELDS

- POSSIBLE INTERACTION TERMS
- S MATRIX
- CROSS SECTION
- EVALUATION OF S-MATRIX ELEMENTS
- WICK'S THEOREM
- PROPAGATORS
- FEYNMAN RULES
- APPLICATION OF WICK'S THEOREM

POSSIBLE INTERACTIONS

CAUSALITY $\Rightarrow$ $\mathcal{L} = \mathcal{L}(\phi(x))$

LORENTZ $\Rightarrow$ $\mathcal{L}$ IS LORENTZ INVARIANT

RENORMALISABILITY :

$[\mathcal{L}] = \text{GeV}^{+4}$ $\quad [\phi] = \text{GeV}^{+1}$ $\quad [\psi] = \text{GeV}^{+3/2}$

$$A \sim g \int^{\Lambda} d^4k$$

$[g] = 1$ RENORMALISABLE

$[g] = \frac{1}{\text{GeV}^n}$ NON RENORMALISABLE

$[g] = \text{GeV}^n$ SUPER RENORMALISABLE

EXAMPLES

$\mu \phi^3$

$\lambda \phi^4$

$g \bar{\psi} \psi \phi$

$e \bar{\psi} \gamma_\mu \psi A^\mu$

$e^2 \phi^* \phi A^\mu A_\mu$

$(A_\mu A^\mu)^2$, $A_\mu A^\mu \partial_\mu A^\mu$

ALL RENORMALISABLE COUPLINGS

GAUGE INVARIANCE

$$\begin{cases} \psi \rightarrow e^{i\Lambda(x)}\psi \\ A_\mu \rightarrow A_\mu - \frac{1}{e}\partial_\mu \Lambda(x) \end{cases}$$

LOCAL PHASE

$$\boxed{\mathcal{D}_\mu = \partial_\mu + ieA_\mu}$$

GAUGE COVARIANT DERIVATIVE

$\Rightarrow$ INTERACTING THEORY

$$\boxed{\mathcal{L}_{QED} = \bar{\psi}(i\gamma\cdot\mathcal{D} - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + |\mathcal{D}_\mu\phi|^2 - m^2|\phi|^2}$$

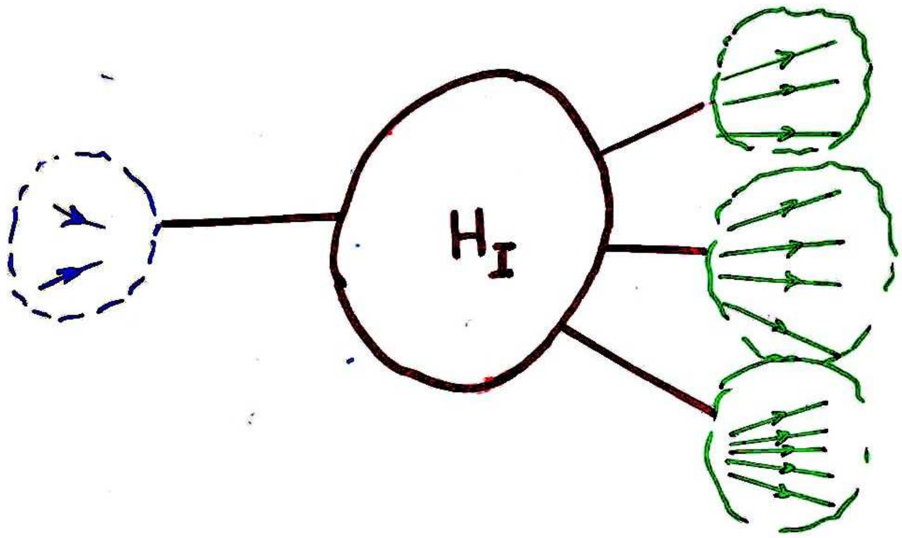
MORE GENERAL:

$$\psi_\alpha \rightarrow \left[e^{i\sum \Lambda_j(x)\,\tau_j(x)}\right]_{\alpha\beta}\psi_\beta$$

MATRICES

$\Rightarrow$ NON-ABELIAN GAUGE THEORIES

S MATRIX : DEFINITION



$|i\rangle \longrightarrow |\phi(t=+\infty)\rangle$

$= |\phi(t=-\infty)\rangle \qquad = \sum S_{fi}\,|f\rangle$

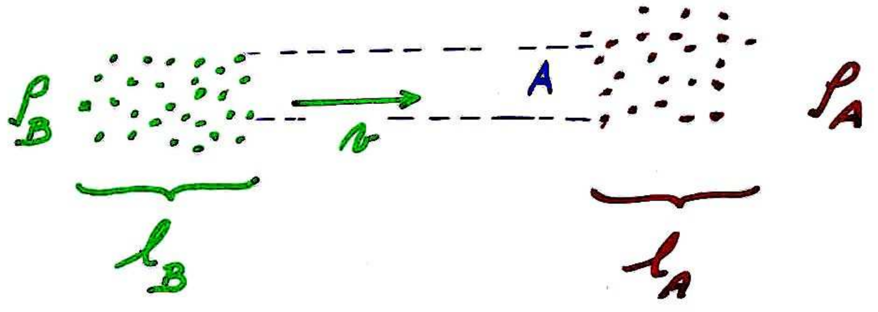
FREE $\qquad = S\,|i\rangle$

S MUST BE UNITARY

$S = \mathbb{1} + iT$

no interaction

SCATTERING CROSS SECTIONS



Number of events $= \rho_A \, l_A \, \rho_B \, l_B \, A \, \sigma = \dfrac{N_A N_B}{A} \sigma$

$$= \sigma \, l_A \, l_B \int d^2x \, \rho_A(x) \rho_B(x)$$

Number of events with specific final state and momenta in $\Delta^3 p_1 \dots \Delta^3 p_n$

$$\frac{dN \prod_i \Delta^3 p_i}{d^3p_1 \dots d^3p_n} = \underbrace{\frac{d\sigma}{d^3p_1 \dots d^3p_n}}_{\text{differential cross section}} \; l_A \, l_B \int d^2x \, \rho_A \rho_B \; \prod \Delta^3 p_i$$

how do we get it from S_{fi}?

$${}_f\langle p_1 p_2 \dots | k_A k_B \rangle_i = \langle p_1 p_2 \dots | S | k_A k_B \rangle$$

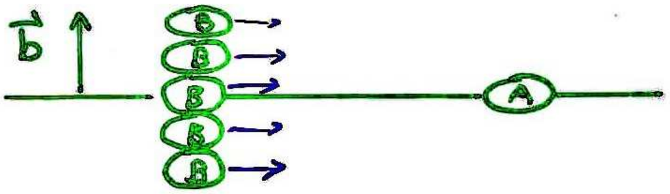
$$\begin{cases} S = \mathbb{1} + iT \\ \text{4-momentum conserved} \end{cases}$$

$$\langle p_1 p_2 \dots | iT | k_A k_B \rangle = (2\pi)^4 \delta^{(4)}\Big(k_A + k_B - \sum p_i\Big) \underbrace{i\mathcal{M}}_{\text{INVARIANT AMPL.}}$$

Probability of scattering

$$P(AB \to 12\ldots n) = \prod_f \underbrace{\frac{d^3p_f}{(2\pi)^3\, 2E_f}}_{\text{LIPS}} \left|\langle p_1 \cdots p_n | \phi_A \phi_B \rangle\right|^2$$

INITIAL WAVE PACKETS



centered on p_A p_B

$$|\phi_A \phi_B\rangle = \int \frac{d^3k_A}{(2\pi)^3\sqrt{2E_A}} \int \frac{d^3k_B}{(2\pi)^3\sqrt{2E_A}}\, \phi_A(k_A)\, \phi_B(k_B)\, e^{-i\vec{b}\cdot\vec{k}_B} |k_A k_B\rangle$$

$$N = \int d^2b\; n_B\; P(\vec{b}) \qquad = n_B N_A \sigma$$

PARTICLES / UNIT AREA

$$\Rightarrow d\sigma = \prod \frac{d^3p_f}{(2\pi)^3 2E_f} \int d^2\vec{b} \prod_{i=A,B} \int \frac{d^3k_i\;\phi_i(k_i)}{(2\pi)^3\sqrt{2E_i}} \int \frac{d^3k_i'\;\phi_i^*(k_i')}{(2\pi)^3\sqrt{2E_i'}}$$

$$\underbrace{e^{i\vec{b}\cdot(\vec{k}_B'-\vec{k}_B)}}_{(2\pi)^2\,\delta^{(2)}(\vec{k}_B'-\vec{k}_B)} \quad \underbrace{\langle p_f | k_i \rangle}_{i\mathcal{M}(2\pi)^4\,\delta^{(4)}(k_i - p_f)} \underbrace{\langle k_i' | p_f \rangle}_{-i\mathcal{M}'^*(2\pi)^4\,\delta^{(4)}(k_i' - p_f)}$$

$$\int dk_A'^z\, dk_B'^z\; \delta(k_A^z + k_B^z - \Sigma p_f^z)\; \delta(E_A' + E_B' - \Sigma E_f)$$

$$= \frac{1}{\left|\frac{k_A^z}{E_A} - \frac{k_B^z}{E_B}\right|} = \frac{1}{v_A - v_B}$$

$$\boxed{d\sigma = \frac{1}{2E_A\, 2E_B\, |v_A - v_B|} \prod_f \frac{d^3p_f}{(2\pi)^3\, 2E_f} \left|\mathcal{M}(p_A + p_B \to \{p_f\})\right|^2 (2\pi)^4\, \delta^{(4)}\left(p_A + p_B - \sum p_f\right)}$$

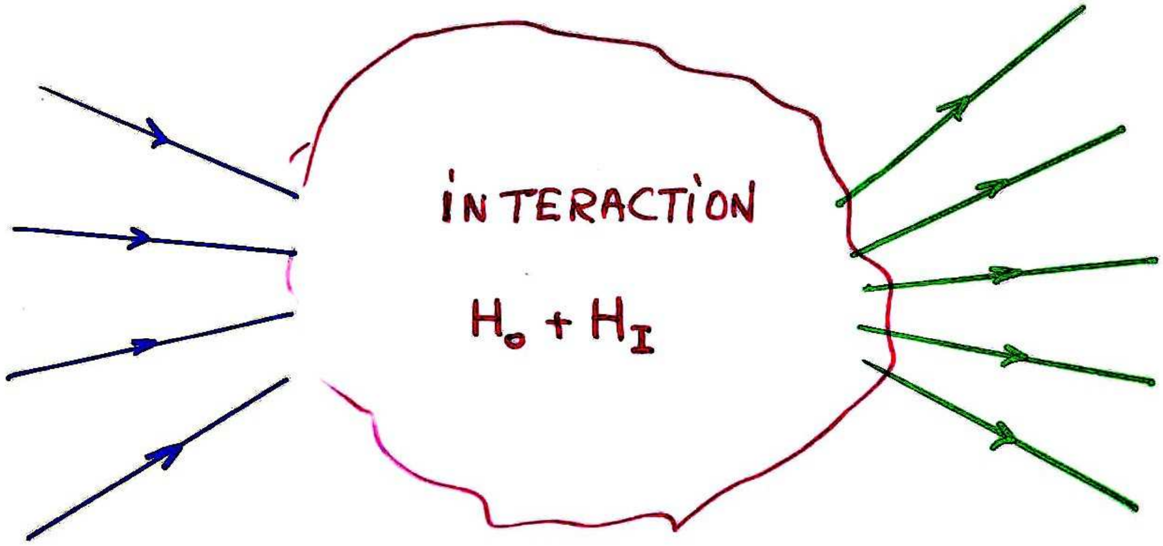
SHOW THIS

2 → 2 : CM frame

$$\frac{d\sigma}{d\Omega^*} = \frac{1}{4E_A E_B |v_A - v_B|} \frac{|\vec{p}|}{(2\pi)^2\, 4E_{cm}} |\mathcal{M}|^2$$

$$\frac{d\sigma}{d\Omega^*} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{CM}^2}$$ (identical masses)

INTERACTIONS



FREE PARTICLES
$|in\rangle$

H_0

FREE PARTICLES
$|out\rangle$

H_0

INTERACTION PICTURE

$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I$

SCHRÖDINGER

$$i\partial_t|\psi\rangle = H|\psi\rangle$$

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$$

INTERACTION

$$|\psi_I(t)\rangle = e^{-iH_0t}|\psi(0)\rangle$$

$$O_I(t) = e^{iH_0t}O_S\,e^{-iH_0t}$$

$|\psi_I\rangle$ CHANGES BECAUSE OF INTERACT.

$$\left\{\begin{array}{l} i\partial_t O_I = [O_I, H_0] \\ i\partial_t|\psi_I\rangle = H_I|\psi_I\rangle \end{array}\right.$$

S MATRIX : EVALUATION

$$|\phi(-\infty)\rangle = |i\rangle$$

$$|\phi(+\infty)\rangle = S|i\rangle$$

PROBABILITY $\quad |\langle f|\phi(\infty)\rangle|^2 = |\langle f|S|i\rangle|$

$$\equiv S_{fi}$$

$$|\phi(\infty)\rangle = \sum_f |f\rangle\langle f|\phi(\infty)\rangle = \sum_f |f\rangle S_{fi}$$

UNITARY : $\quad \sum_f |S_{fi}|^2 = 1$

$$i\frac{d}{dt}|\phi(t)\rangle = H_I(t)|\phi(t)\rangle$$

$$|\phi(t)\rangle = |i\rangle + \frac{1}{i}\int_{-\infty}^{t} dt_1\, H_I(t_1)|\phi(t_1)\rangle$$

$$\simeq |i\rangle + \frac{1}{i}\int_{-\infty}^{t} dt_1\, H_I(t_1)|i\rangle$$

$$+ \frac{1}{(i)^2}\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2\, H_I(t_1) H_I(t_2)|\phi(t_2)\rangle$$

$t \to +\infty$:

$$S = \sum_0^{\infty} \frac{1}{i^n}\int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \int_{-\infty}^{t_2} dt_3 \ldots\ H_I(t_1) H_I(t_2) H_I(t_3)\ldots$$

$$= \sum_0^{\infty} \frac{(-i)^n}{n!}\int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 \int_{-\infty}^{+\infty} dt_3 \ldots T\left(H_I(t_1) H_I(t_2) H_I(t_3) \ldots \right)$$

$$\boxed{\begin{aligned} S &= \sum_0^\infty \frac{(-i)^n}{n!} \int \dots \int d^4x_1 \dots d^4x_n \; T\left(:\mathcal{H}_I(x_1):\dots:\mathcal{H}_I(x_n):\right) \\ &\equiv T \exp\left(-i \int d^4x :\mathcal{H}_I(x):\right) \end{aligned}}$$

SPLIT THE FIELDS:

$$\psi = \psi^+ + \psi^-$$
$$A = A^+ + A^-$$
$$\phi = \phi^+ + \phi^-$$

ϕ^+: annihil. $\quad$ ϕ^-: creation

$$:\psi\psi': = \psi^+\psi'^+ + \psi^-\psi'^- + \psi^-\psi'^+ - \psi'^-\psi^+$$

$$:\phi\phi': = \phi^+\phi'^+ + \phi^-\phi'^- + \phi^-\phi'^+ + \phi'^-\phi^+$$

$$\begin{aligned} AB - :AB: &= \{A^+, B^-\} && \textit{fermions} \\ &= [A^+, B^-] && \textit{bosons} \\ &= \langle 0|AB|0\rangle \end{aligned}$$

$$\begin{aligned} T\left(A(x_1)B(x_2)\right) - :A(x_1)B(x_2): &= \langle 0|T(AB)|0\rangle \\ &\equiv \underbrace{A(x_1)\;B(x_2)} \end{aligned}$$

$$\begin{aligned} T(ABC\dots WXYZ) = {} & :A\dots Z: \\ & + :\underbrace{AB}\dots Z: + :A\underbrace{B\dots Z}: + \dots \\ & + :\underbrace{AB}\dots\underbrace{YZ}: + :\underbrace{A\overbrace{B\dots Y}}\!\!\overbrace{}^{}Z: + \dots \\ & + \dots \end{aligned}$$

$$T\left(:AB\ldots F:_{x_1} \; :xy\ldots z:_{x_2} \right)$$

$$= \; :AB\ldots F: \; :xy\ldots z: \; + \ldots$$

NO EQUAL TIMES

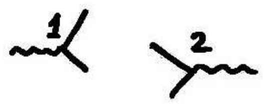
EXAMPLE

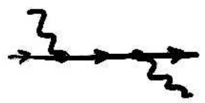
$$\mathcal{H}_I(x) = -e \; :\bar{\psi}(x)\, \gamma\cdot A(x)\, \psi(x):$$

$S^{(1)}$ FORBIDDEN BY 4-MOMENTUM CONSERVATION

$$S^{(2)} = -\frac{e^2}{2!} \int d^4x_1\, d^4x_2 \; \mathcal{S}$$

$$\mathcal{S} = \; :(\bar{\psi} A\cdot\gamma\psi)_1 (\bar{\psi} A\cdot\gamma\psi)_2:$$



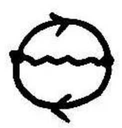


$$+ \; :(\bar{\psi} A\cdot\gamma\psi)_1 (\bar{\psi} A\cdot\gamma\psi)_2: \; + \; :(\bar{\psi} A\cdot\gamma\psi)_1 (\bar{\psi} A\cdot\gamma\psi)_2:$$

$$+ \; :(\bar{\psi} A\cdot\gamma\psi)_1 (\bar{\psi} A\cdot\gamma\psi)_2:$$

$$+ \; :(\bar{\psi} A\cdot\gamma\psi)_1 (\bar{\psi} A\cdot\gamma\psi)_2: + :(\bar{\psi} A\cdot\gamma\psi)_1 (\bar{\psi} A\cdot\gamma\psi)_2:$$

$$+ \; :(\bar{\psi} A\cdot\gamma\psi)_1 (\bar{\psi} A\cdot\gamma\psi)_2:$$



$$+ \; :(\bar{\psi} A\cdot\gamma\psi)_1 (\bar{\psi} A\cdot\gamma\psi)_2:$$

PROPAGATORS

$$\langle 0 | T\, \phi(x)\, \phi(y) | 0 \rangle = D_F(x-x')$$

IS A SOLUTION OF $(\Box + m^2) D = -i\delta^{(4)}(x-y)$

show this

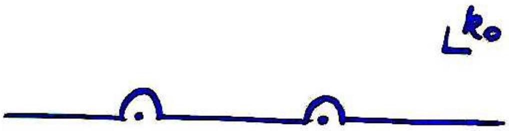
GENERAL SOLUTION : FOURIER $D(x) = \int \frac{d^4 p}{(2\pi)^4} D(k)\, e^{-ik(x)}$

$$(-k^2 + m^2)\, D(k) = -i$$

$$D(k) = \frac{i}{k^2 - m^2}$$

FOURIER BACK: 2 POLES AT $k_0 = \pm\sqrt{m^2 + \vec{k}^2}$

→ PRESCRIPTION:



$$\int \frac{d^4 k}{(2\pi)^4}\, \frac{i}{(k_0-\omega)(k_0+\omega)}\, e^{-ik_0 t}\, e^{i\vec{k}\cdot\vec{x}}$$

RETARDED $= 0$ IF $t < 0$

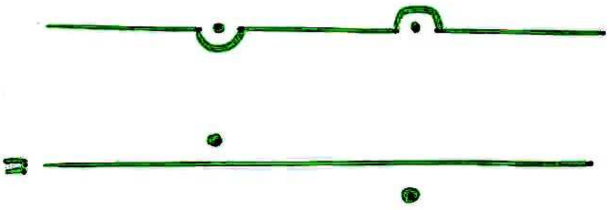
$= 0$ IF $t > 0$

ADVANCED

FEYNMAN

particles $k_0 > 0 \Rightarrow t > 0$

antiparticles $k_0 < 0 \Rightarrow t < 0$



$$\frac{i}{p^2-m^2+i\varepsilon}$$

ALL FIELDS OBEY $(\Box + m^2)\cdots = 0$

$\Rightarrow$ ALWAYS THE SAME DENOMINATOR

$\hat{O}\,\psi = 0$ FREE

$\hat{O}\,D = -i\,\delta^{(4)}$ PROPAGATOR

$\longrightarrow$ $\langle 0|\psi(x)\bar{\psi}(y)|0\rangle = S_{\alpha\beta}(x-y)$

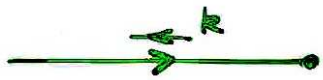
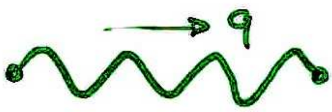
$$S_{\alpha\beta}(k) = \frac{i(\gamma . k + m)_{\alpha\beta}}{k^2-m^2+i\varepsilon}$$

$\langle 0|A_\mu(x)A_\nu(y)|0\rangle = D_{\mu\nu}(x-y)$

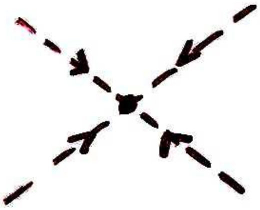
$$D_{\mu\nu}(k) = \frac{-i\,g_{\mu\nu}}{k^2+i\varepsilon}$$ (Feynman gauge)

CALCULATE $D_{\mu\nu}$ IN A GENERAL COVARIANT GAUGE

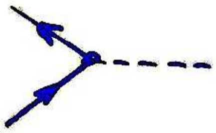
Feynman Rules for $i\mathcal{M}$:

SCALARS		$\dfrac{i}{p^2 - m^2 + i\varepsilon}$
$\overbracket{\phi\,\|q}\rangle$		1
FERMIONS		$\dfrac{i(\not{p} + m)}{p^2 - m^2 + i\varepsilon}$
$\overbracket{\psi\,\|p},s\rangle$		$u^s(p)$ fermion
$\langle \overbracket{p,s\,\|\bar{\psi}}$		$\bar{u}^s(p)$ fermion
$\langle \overbracket{k,s\,\|\psi}$		$v^s(k)$ antifermion
$\overbracket{\bar{\psi}\,\|k},s\rangle$		$\bar{v}^s(k)$ antifermion
PHOTONS		$\dfrac{-i g_{\mu\nu}}{q^2 + i\varepsilon}$
$\overbracket{A_\mu\,\|p}\rangle$		$\varepsilon_\mu(p)$
$\langle \overbracket{p\,\|A_\mu}$		$\varepsilon^*_\mu(p)$

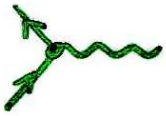
Vertices



$-i\lambda$ $\qquad$ $\mathcal{L}_I = \frac{\lambda}{4!}\phi^4$



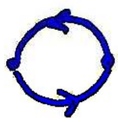
$-ig$ $\qquad$ $\mathcal{L}_I = g\bar{\psi}\psi\phi$



$-ie\gamma^\mu$ $\qquad$ $\mathcal{L}_I = -Q\bar{\psi}\gamma_\mu A^\mu\psi$

(electron has $Q = -e$)

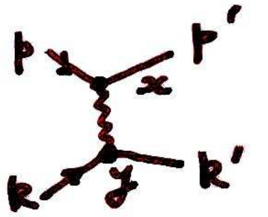
(-1) per fermion loop



EXPLICIT EXAMPLE

$|i\rangle = |e\mu\rangle \qquad p, k$

$|f\rangle = |e\mu\rangle \qquad p', k'$



$$S_{fi} = \langle p'k'|\, T\Big(\frac{1}{2!}(-i)^2 \int e :\bar{\psi}_e \gamma^\mu \psi_e A_\mu: d^4x \int e :\bar{\psi}_\mu \gamma^\nu \psi_\mu A_\nu : d^4y\, |pk\rangle$$

$$\psi_{(x)}|pk\rangle = \underbrace{\int \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{p'}}}\sum_{s'} a^{s'}_{p'} u^{s'}(p') e^{-ip'x}}_{\psi^+} \underbrace{\sqrt{2E_p}\, a^{s\dagger}_p |k\rangle}_{|pk\rangle}$$

$$= e^{-ipx}\, u^s(p)\,|k\rangle \quad = \quad \overbrace{\psi(x)\;|p}k\rangle$$

$$\langle p'k'|\bar{\psi}(x) \;= \langle k'| e^{ip'x}\, \bar{u}^s(p') \;=\; \langle \overbrace{p'k'|\bar{\psi}}$$

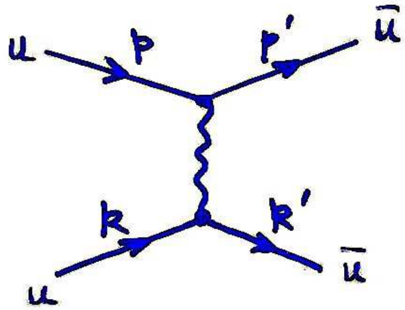
$$\Rightarrow S_{fi} = \langle p'k'|\, T\Big(-\frac{e^2}{2}\int \bar{\psi}\gamma\cdot A\psi\, d^4x \int \bar{\psi}\gamma\cdot A\psi\, d^4y\; |pk\rangle$$

$$= \frac{i^2e^2}{2}\int d^4x\, d^4y\; \bar{u}(p')\gamma^\mu u(p)\, e^{-i(p-p')x}\quad S^A_{\mu\nu}(x-y)\; \bar{u}(k)\gamma^\nu u(k)\; e^{-i(k-k')y}$$

$$= \frac{i^2e^2}{2}\int d^4x\, d^4y\; \bar{u}\gamma^\mu u \left[\int \frac{d^4q}{(2\pi)^4}\frac{-i g_{\mu\nu}}{q^2+i\epsilon}\, e^{-iq(x-y)}\right] \bar{u}\gamma^\nu u \; e^{-i(p-p')x}\, e^{-i(k-k')y}$$

$$= \frac{i^2e^2}{2}\int d^4q \underbrace{\int d^4x\, e^{-i(p-p'+q)}}_{(2\pi)^4\delta^{(4)}(p-p'+q)} \underbrace{\int d^4y\, e^{-i(k-k'-q)}}_{(2\pi)^4\delta^{(4)}(k-k'-q)} \left[\bar{u}\gamma^\mu u \frac{-ig_{\mu\nu}}{q^2+i\epsilon}\bar{u}\gamma^\nu u\right]$$

$$= (2\pi)^4\delta^{(4)}(p+k-p'-k')\underbrace{\frac{i^2e^2}{2}\left[\bar{u}\gamma^\mu u\frac{-ig_{\mu\nu}}{(p-p')^2+i\epsilon}\bar{u}\gamma^\nu u\right]}_{+i\mathcal{M}/2}\quad : \quad x \leftrightarrow y$$



$$= \bar{u}(p') \, -ie\gamma^\mu \, u(p)$$

$$\times \frac{-i\, g_{\mu\nu}}{q^2 + i\varepsilon}$$

$$\times \, \bar{u}(k') \, -ie\gamma^\nu \, u(k)$$

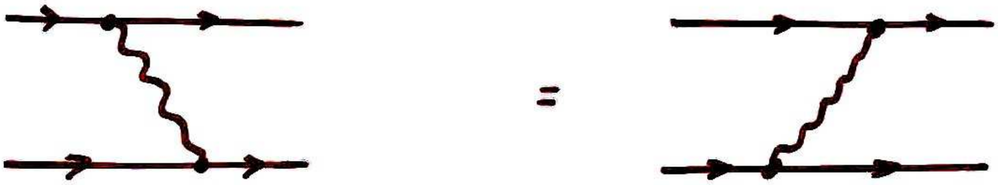
$$= -e^2 \, \bar{u}\gamma^\mu u \frac{-ig_{\mu\nu}}{q^2+i\varepsilon} \, \bar{u}\gamma^\nu u$$

$$= +i\mathcal{M}$$

✓

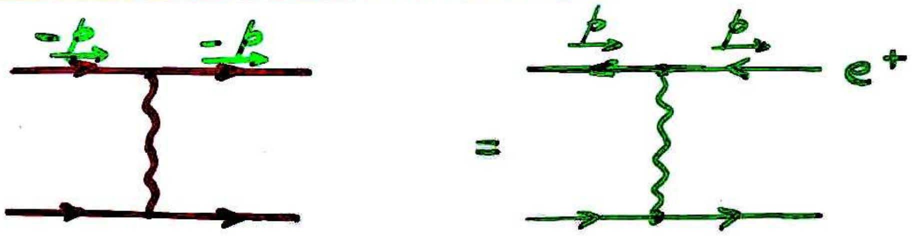
REMARKS

NO TIME AXIS :

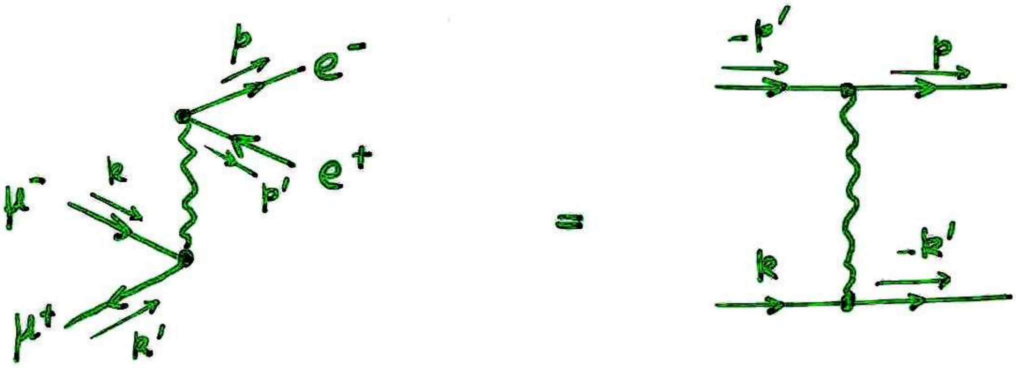


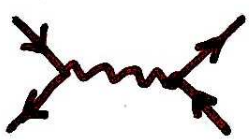
BOTH ORDERINGS IN PROPAGATORS

CROSSING SYMETRY



e^- WITH MOMENTUM $-p$ $= e^+$





⚠ FOR $\mathcal{M}$, NOT $d\sigma$!