3. QUANTIZATION

- RELATIVISTIC WAVE EQUATIONS
  - KLEIN GORDON
  - DIRAC
  - LAGRANGIANS
- SECOND QUANTIZATION
  - FOCK SPACE
  - RELATIVISTIC BOSON FIELD
  - RELATIVISTIC NORMALISATION

QUANTIZATION

a) PROMOTE p and q
   TO $\hat{p}$ and $\hat{q}$

b) ACT ON A HILBERT SPACE

c) $\{\,,\} \rightarrow i[\,,]$

EXAMPLE : COORDINATE REPRESENTATION

$q^\mu = (t, \vec{x})$ $\quad p^\mu = (i\partial_t, -i\vec{\nabla}) = i\partial^\mu$

$E = \frac{1}{2m} p^2 \quad \rightarrow \quad i\partial_t \psi = -\frac{1}{2m} \Delta \psi$

HILBERT SPACE :

SCALAR PRODUCT $\int d^3x\, \varphi^* \psi = \langle \varphi | \psi \rangle$

PROB. INTERP. $\begin{cases} \text{POSITIVE DEFINITE} & \int d^3x\, |\psi|^2 > 0 \\ \text{CONSERVED CURRENT} & \end{cases}$

$$j_\mu = \left(|\psi|^2, -\frac{i}{2m} \psi^* \vec{\nabla} \psi + c.c. \right)$$

THE KLEIN-GORDON EQUATION

$$E^2 = \hat{\vec{p}}^{\,2} + m^2$$

$$\Rightarrow (\Box + m^2)\,\phi = 0$$

SCALAR PRODUCT:

$$\partial_\mu(\phi^* \partial^\mu \phi - \phi\, \partial^\mu \phi^*) = 0$$

$$\Rightarrow j^\mu = \phi^* i \partial^\mu \phi - \phi\, i \partial^\mu \phi^*$$ SHOW THIS

$$j^0 = \phi^* i \partial_t \phi - \phi\, i \partial_t \phi^*$$ NOT POSITIVE DEFINITE!

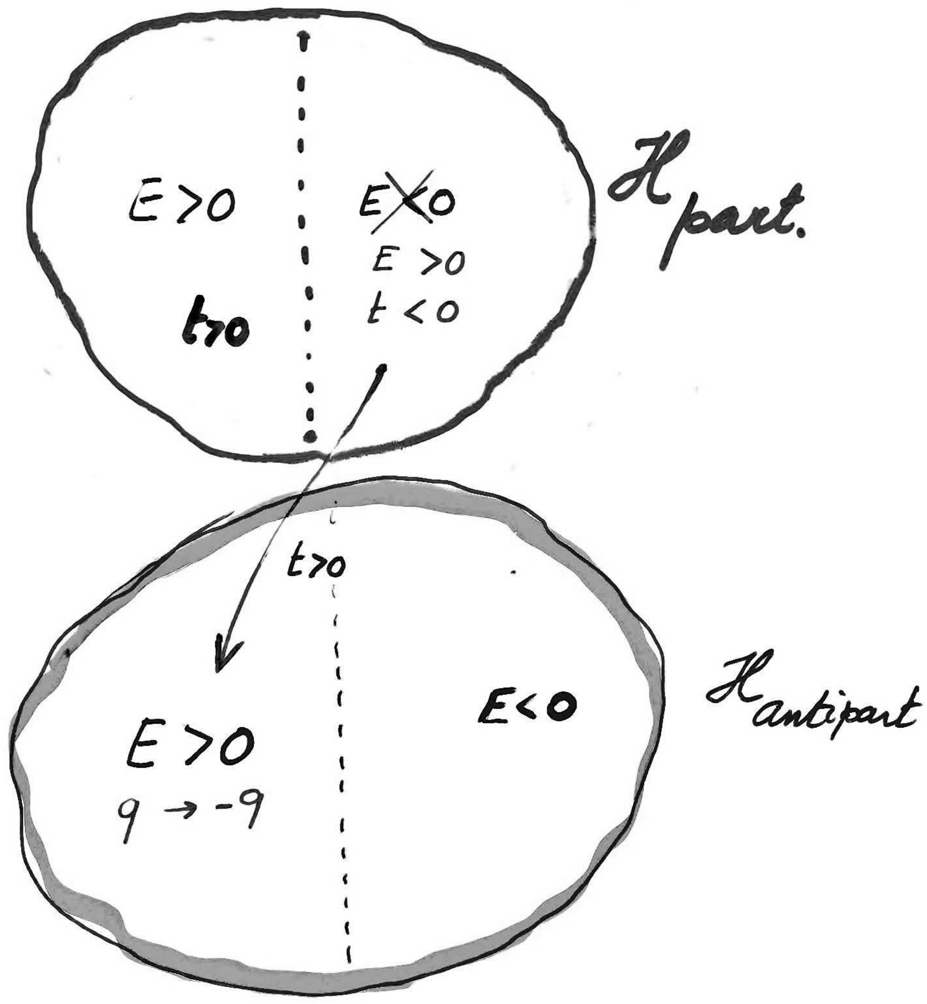
SOLUTIONS: FOR $\vec{p} = 0$:

$$\phi = N e^{\pm imt}$$

$$j^0 = \pm 2m\,|N|^2 \rightarrow$$ CHARGE CURRENT

$\Rightarrow$ NEGATIVE ENERGY STATES HAVE NEGATIVE NORM!

CAN WE LINEARIZE?



$$e^{+i\omega t} = e^{-i\omega(-t)} \qquad \omega = +\sqrt{\vec{p}^{\,2} + m^2}$$

PARTICLE WITH $E<0$ $t>0$

$\Leftrightarrow$ " " $E>0$ $t<0$

$\Leftrightarrow$ ANTIPARTICLE $E>0$ $t>0$

LINEAR FORM 1

$$(\Box + m^2)\,\phi = 0$$

$$\begin{cases} \phi = f + g \\ \frac{1}{m} i\partial_t \phi = f - g \end{cases} \qquad \begin{cases} f = \frac{1}{2}\left(\phi + \frac{i}{m}\partial_t \phi\right) \\ g = \frac{1}{2}\left(\phi - \frac{i}{m}\partial_t \phi\right) \end{cases}$$

$$\Rightarrow \quad \rho = \int (i\phi^* \partial_t \phi - i\phi\,\partial_t \phi^*)\, d^3x$$

$$= m \int [(f^* + g^*)(f - g) - (f + g)(g^* - f^*)]\, d^3x$$

$$= 2m \int (f^* f - g^* g)\, d^3x$$

$$\begin{cases} i\partial_t f = -\frac{1}{2m}\nabla^2 (f+g) + m f \\ i\partial_t g = \frac{1}{2m}\nabla^2(f+g) - m g \end{cases}$$

$$\frac{1}{2} i\partial_t \phi - \frac{1}{2m}\partial_t^2 \phi = \frac{m}{2}(f-g) - \frac{1}{2m}\left(\nabla^2(f+g) - m^2(f+g)\right)$$

$$= \frac{m}{2}(f - g + f + g) - \frac{1}{2m}\nabla^2(f+g)$$

invent $\quad \psi \equiv \begin{pmatrix} f \\ g \end{pmatrix}$

$$\rho = 2m\, \psi^\dagger \sigma_3 \psi$$

$$i\partial_t \psi = \underbrace{\left[\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \frac{\nabla^2}{2m} + m\,\sigma_3\right]}_{\hat{H}} \psi$$

$$\langle \psi | \psi' \rangle = \int d^3x\, \psi^\dagger \sigma_3 \psi'$$

$$\psi(x) = u(p)\, e^{-i p_\mu x^\mu}$$

$$u_+ = N \begin{pmatrix} E + m \\ m - E \end{pmatrix} \qquad u_- = N \begin{pmatrix} m - E \\ m + E \end{pmatrix}$$

LINEARISATION 2 : SPIN

EQUATION LINEAR IN ∂_t AND $\vec{\nabla}$

$$H = \vec{\alpha} \cdot \hat{\vec{p}} + \beta m$$

$$H^2 = |\hat{\vec{p}}|^2 + m^2 = \sum_{ij} (\alpha_i p_i + \beta m)(\alpha_j p_j + \beta m)$$

$$\Rightarrow \quad \begin{cases} \alpha_i^2 = 1 \qquad \beta^2 = 1 \\ \{\alpha_i, \alpha_j\} = \{\beta, \alpha_i\} = 0 \qquad j \neq i \end{cases}$$

α_i, β are hermitian matrices $\qquad H$

of even dimension $\qquad \det \alpha_i \alpha_j = (-)^d \det \alpha_j \alpha_i$

of eigenvalues ± 1 $\qquad \alpha_i^2 = 1$

Pauli matrices

$$\begin{pmatrix} \cdot & 1 \\ 1 & \cdot \end{pmatrix} \begin{pmatrix} \cdot & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & \cdot \\ \cdot & -1 \end{pmatrix}$$

$\Rightarrow$ Weyl equation : $\quad i\partial_t \psi = \pm \vec{\sigma} \cdot \vec{p}\, \psi$

OK for massless fermions

$d = 4$:

$$\vec{\alpha} = \begin{pmatrix} -\vec{\sigma} & \cdot \\ \cdot & \vec{\sigma} \end{pmatrix} \qquad \beta = \begin{pmatrix} \cdot & I \\ I & \cdot \end{pmatrix} \qquad \text{Weyl}$$

$$\vec{\alpha} = \begin{pmatrix} \cdot & \vec{\sigma} \\ \vec{\sigma} & \cdot \end{pmatrix} \qquad \beta = \begin{pmatrix} I & \cdot \\ \cdot & -I \end{pmatrix} \qquad \text{Dirac}$$

Show that these two representations are equivalent

Covariant form of the equation :

$$\gamma_0 \equiv \beta \qquad \vec{\gamma} \equiv \beta \vec{\alpha} \qquad \vec{\gamma}_W = \begin{pmatrix} \cdot & \vec{\sigma} \\ -\vec{\sigma} & \cdot \end{pmatrix}$$

$$\Rightarrow \boxed{(i \partial_\mu \gamma^\mu - m)\, \psi = 0}$$

$$\vec{\gamma}_D = \vec{\gamma}_W$$

$$\boxed{\{ \gamma^\mu, \gamma^\nu \} = 2 g^{\mu\nu}}$$

$$(\gamma^\mu)^+ = \gamma_0 \gamma^\mu \gamma_0$$

Current :

$$\boxed{j^\mu = \psi^+ \gamma_0 \gamma^\mu \psi}$$

$$\bar{\psi} \equiv \psi^+ \gamma_0$$

in particular: $\rho = \psi^+ \psi$ positive

NB: FOR QUANTUM FIELDS $\langle \rho \rangle$ CAN BE < 0

SOLUTIONS :

$$\omega = \sqrt{\vec{p}^2 + m^2} \qquad \chi_+ \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_- \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\psi = u \, e^{-i p \cdot x}$$

$$\Rightarrow \quad E = +\omega$$

$$u = \begin{pmatrix} \chi \\ \frac{\vec{\sigma} \cdot \vec{p}}{\omega + m} \chi \end{pmatrix} \qquad \text{IN DIRAC REP.}$$

$$E = -\omega$$

$$u = \begin{pmatrix} -\frac{\vec{\sigma} \cdot \vec{p}}{\omega + m} \chi \\ \chi \end{pmatrix}$$

PROPERTIES :

$$[H, \vec{\Sigma} \cdot \vec{p}] = 0 \qquad \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & \cdot \\ \cdot & \vec{\sigma} \end{pmatrix}$$

$$[H, \vec{L} + \tfrac{1}{2} \vec{\Sigma}] = 0 \qquad \text{show this.}$$

$\Rightarrow$ TOTAL ANGULAR MOMENTUM

$$\vec{J} = \vec{L} + \frac{\vec{\Sigma}}{2}$$

LORENTZ TRANSFORMATIONS

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$$

$$x_{\mu} x^{\mu} = x'_{\mu} x'^{\mu} \Rightarrow \Lambda_{\mu}{}^{\nu} \Lambda^{\mu}{}_{\tau} = \delta^{\nu}_{\tau}$$
$$(\Lambda^{-1})_{\tau}{}^{\mu} = \Lambda^{\mu}{}_{\tau}$$

$$\partial'_{\mu} = \Lambda_{\mu}{}^{\nu}\, \partial_{\nu}$$

$$\psi \to S\psi = \psi'$$

$$(i\gamma_{\mu}\partial^{\mu} - m)\psi' = (i\gamma_{\mu}\Lambda^{\mu\nu}\partial_{\nu} - m)\, S\psi$$

$$\gamma^{\mu}\Lambda_{\mu}{}^{\nu} S = S\gamma^{\nu}$$

$$\left.\begin{array}{l} \Lambda = 1 + \varepsilon \\ \boxed{S = I + \Gamma\cdot\varepsilon} \end{array}\right\} \quad (1+\Gamma\cdot\varepsilon)\gamma^{\nu}(1-\Gamma\cdot\varepsilon) = (1+\varepsilon)\gamma$$

$$[\Gamma\cdot\varepsilon, \gamma_{\nu}] = \varepsilon_{\nu}{}^{\mu}\gamma^{\mu}$$

$$\boxed{\Gamma_{\mu\nu} = \frac{1}{8}[\gamma_{\mu}, \gamma_{\nu}]}$$

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix}}_{\Lambda^{-1} = \Lambda^{T}} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \simeq 1 + \underbrace{\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \theta & \cdot \\ \cdot & -\theta & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}}_{\varepsilon}$$

$$S = 1 + \frac{\theta}{8}\, 2\varepsilon^{12}[\gamma_1, \gamma_2] = 1 + \frac{\theta}{4}\left[\begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}\right]$$

$$= 1 + \frac{\theta}{4}\begin{pmatrix} 2i\sigma_3 & \cdot \\ \cdot & 2i\sigma_3 \end{pmatrix} = 1 + \frac{i\theta}{2}\vec{\Sigma}\cdot\vec{n}$$

$$= \text{first order of } \exp\left(\frac{i\theta}{2}\vec{\Sigma}\cdot\vec{n}\right)$$

DISCRETE TRANSFORMATIONS

PARITY $\quad S_P = \gamma^0$ $\quad$ SHOW THIS

CHARGE CONJ.

$$\psi_c = C'\psi^{*} = C(\bar{\psi})^{T}$$

$$C' = i\gamma_2 = \begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}$$

$$u^{(1,2)}(-p^{\mu}) = v^{(2,1)}(p^{\mu})$$

TIME REVERSAL

$$\psi' = T\psi^{*}(t)$$
$$T = i\gamma^1\gamma^3$$

$$\psi_c = i\gamma_2\psi^{*}$$
$$\psi_{cP} = \gamma^0 i\gamma_2\psi^{*}$$
$$\psi_{cPT} = i\gamma_1\gamma_3\,\gamma_0\, i\gamma_2\,\psi$$
$$= \gamma_5\psi$$

LORENTZ TRANSFORMATIONS 2

HOW TO COMBINE SPINORS TO MAKE SCALARS, VECTORS ...

$\bar\psi\psi = \psi^\dagger\gamma_0\psi$	SCALAR
$\bar\psi\gamma_\mu\psi$	VECTOR
$\bar\psi i[\gamma_\mu,\gamma_\nu]\psi$	TENSOR
$\bar\psi\gamma_5\psi$	PSEUDO SCALAR
$\bar\psi\gamma_5\gamma_\mu\psi$	PSEUDOVECTOR

$$\underline{\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3}$$

$\gamma_5^\dagger = \gamma_5$

$\gamma_5^2 = I$

$\{\gamma_5,\gamma_\mu\} = 0$

$\gamma_5 = \begin{pmatrix} \cdot & I \\ I & 0 \end{pmatrix}$ (Dirac rep.)

$\psi = \begin{pmatrix} a \\ b \end{pmatrix}$ $\quad \gamma^0 = \begin{pmatrix} I & \\ & -I \end{pmatrix}$ $\quad \bar\psi = (a^\dagger, -b^\dagger)$

$a^\dagger a - b^\dagger b$ IS A SCALAR

$\vec\gamma = \begin{pmatrix} 0 & \vec\sigma \\ -\vec\sigma & 0 \end{pmatrix}$ $\Rightarrow$ $(a^\dagger a + b^\dagger b, + b^\dagger\vec\sigma a + a^\dagger\vec\sigma b)$ IS A 4-VECTOR

ETC.

LAGRANGIAN

$$\mathcal{L}_D = \bar\psi(i\gamma_\mu\partial^\mu - m)\psi$$

$$\frac{\delta\mathcal{L}}{\delta\bar\psi} \Rightarrow (i\gamma_\mu\partial^\mu - m)\psi = 0$$

$$\frac{\delta\mathcal{L}}{\delta\partial_\mu\psi} - \frac{\delta\mathcal{L}}{\delta\psi} \Rightarrow \bar\psi(i\gamma_\mu\partial^\mu + m) = 0$$

$\rightarrow$ VARY ψ AND $\bar\psi$ INDEPENDENTLY

WEYL SPINORS

$\psi = \begin{pmatrix} a \\ b \end{pmatrix}$

CHIRAL
$$\left\{\begin{aligned} \psi_R &= \tfrac{1}{2}(1+\gamma_5)\psi = \begin{pmatrix} a \\ 0 \end{pmatrix} \\ \psi_L &= \tfrac{1}{2}(1-\gamma_5)\psi = \begin{pmatrix} 0 \\ b \end{pmatrix} \end{aligned}\right.$$
IN WEYL REP.

$$\Rightarrow \bar\psi(i\gamma.\partial - m)\psi$$

$$= \bar\psi_R\, i\gamma.\partial\psi_R + \bar\psi_L\, i\gamma.\partial\psi_L + m(\bar\psi_R\psi_L + \bar\psi_L\psi_R)$$

$\psi_R \longleftrightarrow \psi_L$ UNDER P

$\psi_R \sim$ HELICITY STATE IF $M=0$

SECOND QUANTIZATION : BOSONS

HILBERT SPACE

$\mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots$ FOCK SPACE

ISOMORPHISM BETWEEN FOCK SPACE AND AN ∞ NUMBER OF HARMONIC OSCILLATORS

$$H_{h.o.} = \frac{1}{2}(\hat{p}^2 + q^2) \qquad \begin{cases} [p,p] = [q,q] = 0 \\ [q,p] = i \end{cases}$$

$$= \frac{1}{2}\Big[\underbrace{(\hat{p}+iq)}_{a^+}\underbrace{(\hat{p}-iq)}_{a} + 1\Big]$$

$$\begin{cases} [a,a] = [a^+,a^+] = 0 \\ [a,a^+] = 1 \end{cases}$$

$|0\rangle$ = FUNDAMENTAL STATE :

$$a|0\rangle = 0$$
$$H|0\rangle = \frac{1}{2}|0\rangle$$

$\Rightarrow$ $(a^+)^n|0\rangle$ IS AN EIGENSTATE

$$H(a^+)^n|0\rangle = (a^+)^n(H+n)|0\rangle$$
$$= (n+\tfrac{1}{2})(a^+)^n|0\rangle$$

SEVERAL INDEPENDENT OSCILLATORS

$$H = \frac{1}{2}\sum_a \hat{p}_a^2 + q_a^2 \qquad [a_a^+, a_b^+] = [a_a, a_b] = 0$$
$$|0\rangle = \bigotimes_a |0_a\rangle \qquad [a_a, a_b^+] = \delta_{ab}$$
$$a_a|0\rangle = 0$$

NORMALISATION

$$|A\rangle = \sum \alpha_n a_n^+ |0\rangle$$
$$|B\rangle = \sum \beta_n a_n^+ |0\rangle$$

$$\langle B|A\rangle = \sum \beta_m^* \alpha_n \langle 0|a^m a^{+n}|0\rangle$$
$$= \sum \beta_m^* \alpha_n \langle 0|a^{m-1}a^{+n}a + a^{m-1} n a^{+n-1}|0\rangle$$
$$= \sum \beta_m^* \alpha_n \, n! \, \delta_{mn}$$

FOCK STATES :

FREE STATES $|\alpha^1\rangle \ldots |\alpha^n\rangle$ $\qquad \langle\alpha^a|\alpha^b\rangle = \delta^{ab}$

g PARTICLES $|\psi\rangle = \frac{1}{u!}\sum_{perm} |\alpha_1^a\rangle|\alpha_2^b\rangle\ldots|\alpha_u^g\rangle$

$\langle\psi|\psi\rangle = 1$ IF STATES ARE DIFFERENT

$= n!$ IF n STATES ARE IDENTICAL

$\frac{1}{\sqrt{2}}(|\alpha^1\rangle|\alpha^1\rangle + |\alpha^1\rangle|\alpha^1\rangle) = \sqrt{2}\,|\alpha^1\rangle|\alpha^1\rangle \rightarrow$ NORM 2

$= n_1! \, n_2! \ldots n_n!$

$\boxed{N = \sum a_i^+ a_i}$ = TOTAL NUMBER

$$a_\beta |\ldots n_\beta \ldots\rangle = \sqrt{n_\beta}\,|\ldots, n_{\beta-1}, \ldots\rangle$$
$$a_\beta^+ |\ldots n_\beta \ldots\rangle = \sqrt{n_\beta+1}\,|\ldots, n_{\beta+1}, \ldots\rangle$$

$|\alpha^a\rangle$

$\sqrt{n!}\ \left(|\alpha^a\rangle|\alpha^a\rangle\dots|\alpha^a\rangle\right)$

3 BOSONS IN $|\alpha^3\rangle$

1 BOSON IN $|\alpha^2\rangle$

2 BOSONS IN $|\alpha^1\rangle$

INTERACTION:

$V(\vec{z})$ $\quad \langle\alpha^i|V|\alpha^j\rangle = V^{ij}$

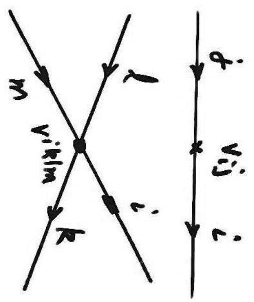
$V(\vec{z}_i-\vec{z}_j)$ $\quad \langle\alpha^i\alpha^k|V|\alpha^l\alpha^m\rangle = V^{iklm}$

$a_a^+\,|0\rangle$

$(a_a^+)^n\,|0\rangle$

$V = \sum V^{ij}\, a_i^+ a_j$

$V = \sum V^{iklm}\, a_i^+ a_k^+ a_l a_m$



LORENTZ TRANSFORMATIONS

$x' = \Lambda x + a$

$|\phi'\rangle = U(\Lambda,a)\,|\phi\rangle$

$\uparrow$

IRREDUCIBLE REPRESENTATION OF SO(3,1)
MASS m, SPIN 0

$\langle x|U(\Lambda,a)|\phi\rangle = \langle x|\phi'\rangle = \langle\Lambda^{-1}(x-a)|\phi\rangle$

$\langle k|\phi'\rangle = e^{ik.a}\,\langle\Lambda^{-1}k|\phi\rangle$

N-PARTICLE STATES:

$\langle k_1\dots k_N|U(\Lambda,a) = e^{i\sum k_i.a}\,\langle\Lambda^{-1}k_1\dots\Lambda^{-1}k_N|$

$\langle k_1\dots k_n|U(\Lambda,a)\,a_k|\phi\rangle$

$= \langle\Lambda^{-1}k_1\dots\Lambda^{-1}k_n|\,a_k\,|\phi\rangle\, e^{i\sum k_i a}$

$= \sqrt{n_k+1}\ \langle k,\Lambda^{-1}k_1\dots\Lambda^{-1}k_n|\phi\rangle\, e^{i\sum k_i a}$

$= \langle k_1\dots k_n|e^{-i\Lambda k.a}\ a(\Lambda k)\ U(\Lambda,a)\,|\phi\rangle$

$= \langle k_1\dots k_n|\,U(\Lambda,a)\,U^{-1}(\Lambda,a)\,a(\Lambda,k)\ U(\Lambda,a)\,|\phi\rangle\, e^{-i\Lambda k.a}$

$$\Rightarrow \boxed{\begin{cases} U(\Lambda,a)\,a_k\,U(\Lambda,a)^{-1} = e^{-i\Lambda k.a}\,a(\Lambda k) \\ U(\Lambda,a)\,a_k^+\,U(\Lambda,a)^{-1} = e^{i\Lambda k.a}\,a^+(\Lambda k)\end{cases}}$$

FIELDS

CAN WE DEFINE OPERATORS THAT TRANSFORM LIKE SCALARS, I.E.

$$U \phi(x) U^{-1} = \phi(\Lambda x + a) \quad ?$$

$$\boxed{\phi^{(+)}(x) = \int_{k_0>0} \frac{d^3k}{k_0} \frac{1}{\sqrt{2(2\pi)^3}} e^{-ik.x} a_{\vec{k}}}$$

$$(\Box + m^2)\phi^+(x) = 0 \qquad k_0 = \sqrt{\vec{k}^2 + m^2}$$

$$\boxed{\phi^-(x) = \int_{k_0>0} \frac{d^3k}{k_0} \frac{1}{\sqrt{2(2\pi)^3}} e^{+ik.x} a^+_{\vec{k}}}$$

$$U(\Lambda,a)\, \phi^+(x)\, U(\Lambda,a)^{-1} = \int_+ d\Omega(k)\, e^{-ik.x}\, U a_{\vec{k}} U^{-1}$$

$$= \int_+ d\Omega(k)\, e^{-i\Lambda k.\Lambda x}\, e^{-i\Lambda k.a}\, a_{\Lambda\vec{k}}$$

$$= \int_+ d\Omega(k')\, e^{-ik'.(\Lambda x+a)}\, a_{\vec{k}'}$$

$$= \phi^+(\Lambda x + a)$$

SIMILARLY FOR ϕ^-

HERMITIAN FIELD

$$\phi(x) = \phi^+(x) + \phi^-(x)$$

$$(\Box + m^2)\phi(x) = 0$$

$$[\phi(x), \phi(x')] = \frac{1}{2(2\pi)^3} \int_{k_0>0} \frac{d^3k}{k_0} \left(e^{-ik(x-x')} - e^{ik.(x-x')}\right)$$

prove this $\rightarrow$

$$\begin{cases} = \dfrac{+i}{(2\pi)^3} \displaystyle\int_{k_0>0} e^{ik(x-x')} \frac{d^3k}{k_0} \sin k_0(x_0 - x_0') \\ = 0 \quad \text{IF} \quad x_0 = x_0' \end{cases}$$

$$\equiv \Delta(x - x')$$

$$\Delta(\vec{x}, t) = \frac{1}{4\pi} \frac{i}{r} \partial_r F(r,t)$$

$$F(r,t) = \begin{cases} J_0(m\sqrt{t^2 - r^2}) & t > r \\ 0 & -r < t < r \\ -J_0(m\sqrt{t^2 - r^2}) & t < -r \end{cases}$$

$$\Delta(x) = -\frac{i}{2\pi} \varepsilon(t) \left\{ \delta(x^2) - \frac{m^2}{2} \theta(x^2) \frac{J_1(m\sqrt{x^2})}{m\sqrt{x^2}} \right\}$$

CONJUGATE FIELD

$$\pi(x) = \partial_t \phi(x)$$

$$= -\frac{i}{\sqrt{2(2\pi)^3}} \int_{k_0>0} d^3k \left(e^{-ik.x} a_{\vec{k}} - e^{ik.x} a_{\vec{k}}^{+}\right)$$

$$\boxed{\left[\pi(x), \phi(x')\right]_{t=t'} = -i\,\delta^{(3)}(\vec{x}-\vec{x}\,')}$$

prove this

$\Rightarrow$ WE RECOVER THE LAGRANGIAN COMMUTATORS!

RELATIVISTIC NORMALISATION

$$|\vec{p}\rangle = C\, a_{\vec{p}}^{+} |0\rangle$$

$$\langle \vec{p} | \vec{q} \rangle = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})\, C^2$$

$\xrightarrow{\Lambda}$ $|p'\rangle$

$$\langle p' | q' \rangle = (2\pi)^3 \delta^{(3)}(\vec{p}\,'-\vec{q}\,')\, C^2$$

$$\int d^3p'\, \delta^{(3)}(\vec{p}\,'-\vec{q}\,') = \int \frac{p_0'}{p_0} \delta^{(3)}(\vec{p}\,'-\vec{q}\,')\, d^3p$$

$\frac{d^3p}{p_0}$ invariant $\rightarrow$ replace d^3p with this

$$\Rightarrow p^0 \delta^{(3)}(\vec{p}-\vec{q}) = p_0' \delta^{(3)}(\vec{p}\,'-\vec{q}\,')$$

ABSORB THIS IN THE NORMALISATION

$$|p\rangle = \sqrt{2E_p}\, a_{\vec{p}}^{+} |0\rangle \qquad [|p\rangle = GeV^{-1}]$$

$$\langle p | q \rangle = (2\pi)^3\, 2E_p\, \delta^{(3)}(\vec{p}-\vec{q}) \qquad [\langle p|q\rangle] = GeV^{-2}$$

$$[a_{\vec{p}}, a_{\vec{p}'}^{+}] = (2\pi)^3 \cdot \delta^{(3)}(\vec{p}-\vec{p}\,') \qquad [a] = GeV^{-3/2}$$

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_{\vec{p}} e^{i\vec{p}.\vec{x}} + a_{\vec{p}}^{+} e^{-i\vec{p}.\vec{x}}\right)$$

$$[\phi] = GeV$$

RECIPE

$$S = \int d^4x \, \mathcal{L}$$

- SYMETRIES LEAVE S INVARIANT
- S EXTREMUM
  - → EULER - LAGRANGE FOR FIELDS
- CONJUGATE MOMENTA π

EQUAL TIMES
$$\begin{cases} [\pi, \phi] = i\delta^{(3)} \\ [\pi, \pi] = [\phi, \phi] = 0 \end{cases}$$

- EXPAND IN TERMS OF EIGEN FUNCTIONS
  - → COEFFICIENTS ARE CREATION + ANNIHILATION OPERATORS
  - → HILBERT SPACE KNOWN
- EVOLVE OPERATORS USING H

$$i\partial_t O = [H, O]$$

- ONLY ONE FIELD IN ALL SPACETIME
  - → PARTICLES ARE IDENTICAL