

3. QUANTIZATION

- . RELATIVISTIC WAVE EQUATIONS
 - KLEIN GORDON
 - DIRAC
 - LAGRANGIANS
- . SECOND QUANTIZATION
 - FOCK SPACE
 - RELATIVISTIC BOSON FIELD
 - RELATIVISTIC NORMALISATION

QUANTIZATION

a) PROMOTE p and q
TO \hat{p} and \hat{q}

b) ACT ON A HILBERT SPACE

c) $\{, \}$ \rightarrow $i[,]$

EXAMPLE : COORDINATE REPRESENTATION

$$q^\mu = (t, \vec{x}) \quad p^\mu = (i\partial_t, -i\vec{\nabla}) = i\partial^\mu$$

$$E = \frac{1}{2m} p^2 \quad \rightarrow \quad i\partial_t \psi = -\frac{1}{2m} \Delta \psi$$

HILBERT SPACE:

SCALAR PRODUCT $\int d^3x \psi^* \psi = \langle \psi | \psi \rangle$

POSITIVE DEFINITE $\int d^3x |\psi|^2 > 0$

CONSERVED CURRENT

$$j_\mu = (|\psi|^2, -\frac{i}{2m} \psi^* \vec{\nabla} \psi + c.c.)$$

PROB.
INTERP.

THE KLEIN-GORDON EQUATION

$$E^2 = \hat{p}^2 + m^2$$

$$\Rightarrow (\square + m^2) \phi = 0$$

SCALAR PRODUCT:

$$\partial_\mu (\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*) = 0$$

$$\Rightarrow j^\mu = \phi^* i \partial^\mu \phi - \phi i \partial^\mu \phi^*$$

SHOW
THIS

$$j^0 = \phi^* i \partial_t \phi - \phi i \partial_t \phi^*$$

NOT POSITIVE
DEFINITE!

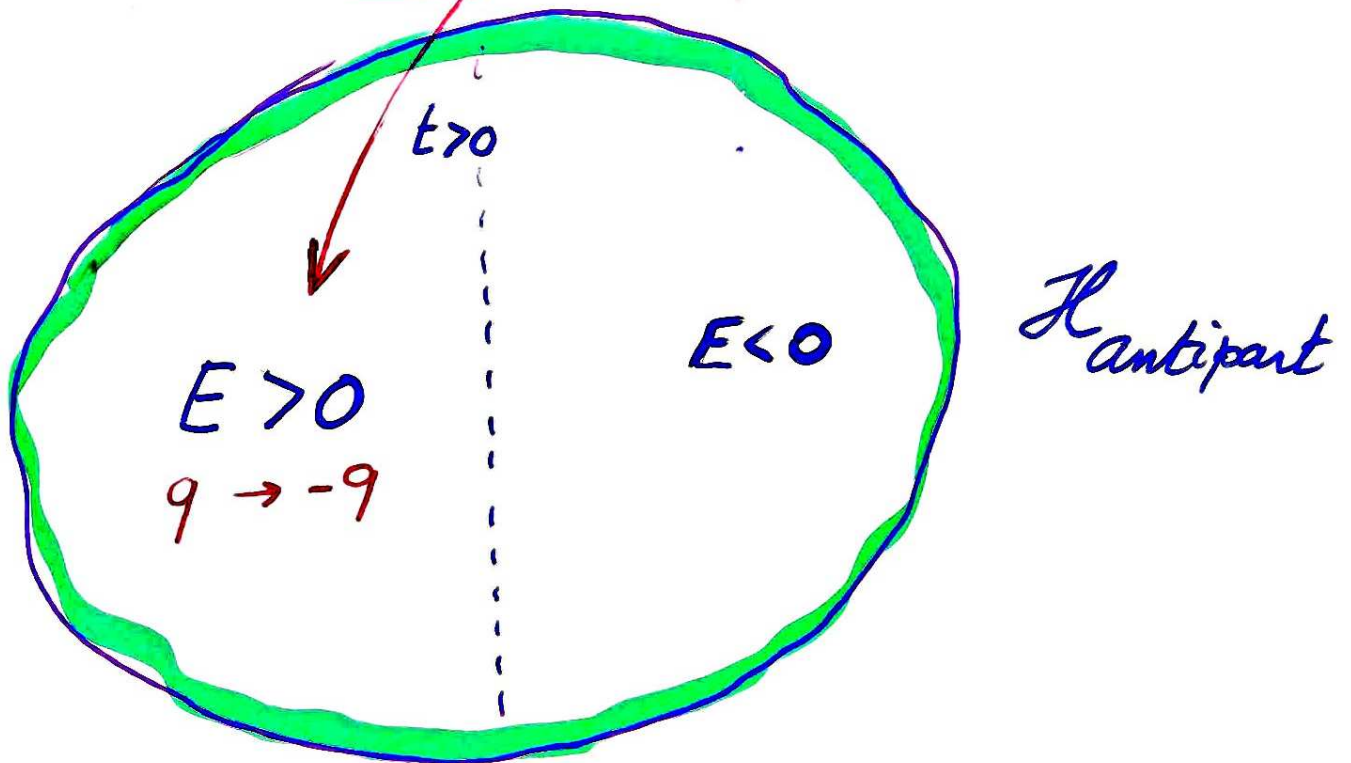
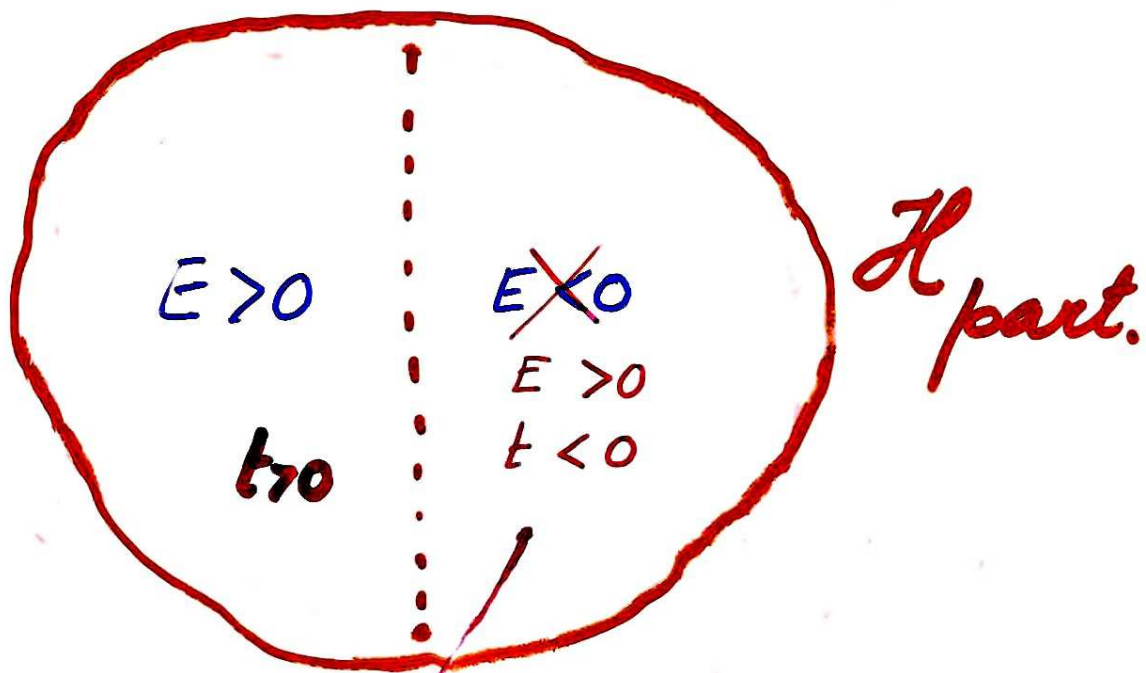
SOLUTIONS: FOR $\vec{p} = 0$:

$$\phi = N e^{\pm i m t}$$

$$j^0 = \pm 2m |N|^2 \rightarrow \text{CHARGE CURRENT}$$

\Rightarrow NEGATIVE ENERGY STATES
HAVE NEGATIVE NORM!

CAN WE LINEARIZE?



$$e^{+i\omega t} = e^{-i\omega(-t)}$$

$$\omega = +\sqrt{\vec{p}^2 + m^2}$$

PARTICLE WITH $E < 0$ $t > 0$

\Leftrightarrow " " $E > 0$ $t < 0$

\Leftrightarrow ANTI PARTICLE $E > 0$ $t > 0$

LINEAR FORM 1

$$(\square + m^2) \phi = 0$$

$$\begin{cases} \phi = f + g \\ \frac{i}{m} \partial_t \phi = f - g \end{cases} \quad \begin{cases} f = \frac{1}{2} (\phi + \frac{i}{m} \partial_t \phi) \\ g = \frac{1}{2} (\phi - \frac{i}{m} \partial_t \phi) \end{cases}$$

$$\begin{aligned} \Rightarrow P &= \int (i \phi^* \partial_t \phi - i \phi \partial_t \phi^*) d^3x \\ &= m \int [(f^* + g^*)(f - g) - (f + g)(g^* - f^*)] d^3x \\ &= 2m \int (f^* f - g^* g) d^3x \end{aligned}$$

$$\begin{cases} i \partial_t f = -\frac{1}{2m} \nabla^2 (f + g) + m f \\ i \partial_t g = \frac{1}{2m} \nabla^2 (f + g) - m g \end{cases}$$

$$\begin{aligned} \frac{1}{2} i \partial_t \phi - \frac{1}{2m} \partial_t^2 \phi &= \frac{m}{2} (f - g) - \frac{1}{2m} (\nabla^2 (f + g) - m^2 (f + g)) \\ &= \frac{m}{2} (f - g + f + g) - \frac{1}{2m} \nabla^2 (f + g) \end{aligned}$$

invent $\psi \equiv \begin{pmatrix} f \\ g \end{pmatrix}$

$$\begin{aligned} P &= 2m \psi^\dagger \sigma_3 \psi \\ i \partial_t \psi &= \underbrace{\left[\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \frac{\nabla^2}{2m} + m \sigma_3 \right]}_{\hat{H}} \psi \end{aligned}$$

$$\langle \psi | \psi' \rangle = \int d^3x \psi^\dagger \sigma_3 \psi'$$

$$\psi(x) = u(p) e^{-i p_\mu x^\mu}$$

$$u_+ = N \begin{pmatrix} E + m \\ m - E \end{pmatrix} \quad u_- = N \begin{pmatrix} m - E \\ m + E \end{pmatrix}$$

LINEARISATION 2 : SPIN

EQUATION LINEAR IN ∂_t AND $\vec{\nabla}$

$$H = \vec{\alpha} \cdot \hat{\vec{p}} + \beta m$$

$$H^2 = |\hat{\vec{p}}|^2 + m^2 = \sum_{ij} (\alpha_i p_i + \beta m)(\alpha_j p_j + \beta m)$$

$$\Rightarrow \begin{cases} \alpha_i^2 = 1 & \beta^2 = 1 \\ \{\alpha_i, \alpha_j\} = \{\beta, \alpha_i\} = 0 & j \neq i \end{cases}$$

α_i, β are hermitian matrices
of even dimension
of eigenvalues ± 1

$$H$$

$$\det \alpha_i \alpha_j = (-1)^d \det \alpha_j \alpha_i$$

$$\alpha_i^2 = 1$$

Pauli matrices

$$\begin{pmatrix} i & 1 \\ 1 & \cdot \end{pmatrix} \begin{pmatrix} i & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & \\ \cdot & -1 \end{pmatrix}$$

$$\Rightarrow \text{Weyl equation} : i \partial_t \psi = \pm \vec{\sigma} \cdot \vec{p} \psi$$

OK for massless fermions

$d=4$:

$$\vec{\alpha} = \begin{pmatrix} \sigma_1 & \\ & \sigma_3 \end{pmatrix} \quad \beta = \begin{pmatrix} & I \\ I & \end{pmatrix} \quad \text{Weyl}$$

$$\vec{\alpha} = \begin{pmatrix} & \sigma_1 \\ \sigma_3 & \end{pmatrix} \quad \beta = \begin{pmatrix} I & \\ & -I \end{pmatrix} \quad \text{Dirac}$$

Show that these two representations are equivalent

Covariant form of the equation :

$$\gamma_0 \equiv \beta \quad \vec{\gamma} \equiv \beta \vec{\alpha} \quad \vec{\gamma}_w = \begin{pmatrix} & \sigma_1 \\ -\sigma_3 & \end{pmatrix}$$
$$\Rightarrow \boxed{(i \partial_\mu \gamma^\mu - m) \psi = 0}$$

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}}$$

$$(\gamma^\mu)^\dagger = \gamma_0 \gamma^\mu \gamma_0$$

$$\vec{\gamma}_D = \vec{\gamma}_w$$

Current :

$$\boxed{j^\mu = \psi^\dagger \gamma_0 \gamma^\mu \psi}$$

$$\bar{\psi} \equiv \psi^\dagger \gamma_0$$

in particular: $p = \psi^\dagger \psi$ positive

NB: FOR QUANTUM FIELDS

$\langle p \rangle$ CAN BE < 0

SOLUTIONS:

$$\omega = \sqrt{\vec{p}^2 + m^2}$$

$$\chi_+ \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \chi_- \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\psi = u e^{-i\vec{p}\cdot\vec{x}}$$

$$\Rightarrow E = +\omega$$

$$u = \begin{pmatrix} \chi \\ \frac{\vec{\sigma}\cdot\vec{p}}{\omega+m} \chi \end{pmatrix}$$

IN DIRAC REP.

$$E = -\omega$$

$$u = \begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{\omega+m} \chi \\ \chi \end{pmatrix}$$

PROPERTIES:

$$[H, \vec{\Sigma}\cdot\vec{p}] = 0 \quad \vec{\Sigma} = \begin{pmatrix} \tau_3 & \\ & \tau_3 \end{pmatrix}$$

$$[H, \vec{L} + \frac{1}{2}\vec{\Sigma}] = 0$$

show this.

\Rightarrow TOTAL ANGULAR MOMENTUM

$$\vec{J} = \vec{L} + \frac{\vec{\Sigma}}{2}$$

LORENTZ TRANSFORMATIONS

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}$$

$$x_{\mu} x^{\mu} = x'_{\mu} x'^{\mu} \Rightarrow \Lambda_{\mu}^{\nu} \Lambda^{\mu}_{\epsilon} = \delta^{\nu}_{\epsilon}$$

$$(\Lambda^{-1})^{\mu}_{\epsilon} = \Lambda^{\mu}_{\epsilon}$$

$$\partial'_{\mu} = \Lambda_{\mu}^{\nu} \partial_{\nu}$$

$$\psi \rightarrow S \psi = \psi'$$

$$(i \gamma_{\mu} \partial^{\mu} - m) \psi' = (i \gamma_{\mu} \Lambda^{\mu\nu} \partial_{\nu} - m) S \psi$$

$$\gamma^{\mu} \Lambda_{\mu}^{\nu} S = S \gamma^{\nu}$$

$$\left. \begin{array}{l} \Lambda = 1 + \epsilon \\ S = I + \Gamma \cdot \epsilon \end{array} \right\} \begin{array}{l} (1 + \Gamma \cdot \epsilon) \gamma^{\nu} (1 - \Gamma \cdot \epsilon) = (1 + \epsilon) \gamma^{\nu} \\ [\Gamma \cdot \epsilon, \gamma_{\nu}] = \epsilon_{\nu}^{\mu} \gamma^{\mu} \end{array}$$

$$\Gamma_{\mu\nu} = \frac{1}{8} [\gamma_{\mu}, \gamma_{\nu}]$$

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & \cos \theta & \sin \theta & \\ & -\sin \theta & \cos \theta & \\ & & & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \approx 1 + \begin{pmatrix} & & \theta & \\ & & -\theta & \\ & & & \\ & & & \end{pmatrix}$$

$\nearrow \Lambda^{-1} = \Lambda^T$
 $\nearrow \epsilon$

$$S = 1 + \frac{\theta}{8} 2 \epsilon^{12} [\gamma_1, \gamma_2] = 1 + \frac{\theta}{4} \left[\begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix} \right]$$

$$= 1 + \frac{\theta}{4} \begin{pmatrix} 2i\sigma_3 & \\ & 2i\sigma_3 \end{pmatrix} = 1 + \frac{i\theta}{2} \vec{\Sigma} \cdot \vec{n}$$

$$= \text{first order of } \exp\left(\frac{i\theta}{2} \vec{\Sigma} \cdot \vec{n}\right)$$

DISCRETE TRANSFORMATIONS

PARITY

$$S_p = \gamma^0$$

SHOW
THIS

CHARGE CONJ.

$$\begin{aligned}\psi_c &= c' \psi^* \\ &= c (\bar{\psi})^T\end{aligned}$$

$$c' = i\gamma_2 = \begin{pmatrix} \dots & \dots & \dots & 1 \\ & & -1 & \\ & & & \\ 1 & & & \end{pmatrix}$$

$$u^{(1,2)}(-p^\mu) = v^{(2,1)}(p^\mu)$$

TIME REVERSAL

$$\psi' = T \psi^*(t)$$

$$T = i\gamma^1\gamma^3$$

$$\psi_c = i\gamma_2 \psi^*$$

$$\psi_{cp} = \gamma^0 i\gamma_2 \psi^*$$

$$\psi_{cPT} = i\gamma_1 \gamma_3 \gamma_0 i\gamma_2 \psi$$

$$= \gamma_5 \psi$$

LORENTZ TRANSFORMATIONS 2

HOW TO COMBINE SPINORS TO
MAKE SCALARS, VECTORS...

$$\bar{\psi}\psi = \psi^\dagger \gamma_0 \psi$$

SCALAR

$$\bar{\psi} \gamma_\mu \psi$$

VECTOR

$$\bar{\psi} i[\gamma_\mu, \gamma_\nu] \psi$$

TENSOR

$$\bar{\psi} \gamma_5 \psi$$

PSEUDO SCALAR

$$\bar{\psi} \gamma_5 \gamma_\mu \psi$$

PSEUDO VECTOR

$$\underline{\gamma_5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3}$$

$$\gamma_5^\dagger = \gamma_5$$

$$\gamma_5^2 = I$$

$$\{\gamma_5, \gamma_\mu\} = 0$$

$$\gamma_5 = \begin{pmatrix} & I \\ I & 0 \end{pmatrix} \quad (\text{Dirac rep.})$$

$$\psi = \begin{pmatrix} a \\ b \end{pmatrix} \quad \gamma^0 = \begin{pmatrix} I & \\ & -I \end{pmatrix} \quad \bar{\psi} = (a^\dagger, -b^\dagger)$$

$$a^\dagger a - b^\dagger b \quad \text{IS A SCALAR}$$

$$\vec{\sigma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \Rightarrow (a^\dagger a + b^\dagger b, +b^\dagger \vec{\sigma} a + a^\dagger \vec{\sigma} b)$$

IS A 4-VECTOR

ETC.

LAGRANGIAN

$$\mathcal{L}_D = \bar{\psi} (i \gamma_\mu \partial^\mu - m) \psi$$

$$\frac{\delta \mathcal{L}}{\delta \bar{\psi}} \Rightarrow (i \gamma_\mu \partial^\mu - m) \psi = 0$$

$$\frac{\delta \mathcal{L}}{\delta \psi} - \frac{\delta \mathcal{L}}{\delta \partial_\mu \psi} \Rightarrow \bar{\psi} (i \gamma_\mu \partial^\mu + m) = 0$$

→ VARY ψ AND $\bar{\psi}$ INDEPENDENTLY

WEYL SPINORS

$$\psi = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\left. \begin{array}{l} \psi_R = \frac{1}{2} (1 + \gamma_5) \psi = \begin{pmatrix} a \\ 0 \end{pmatrix} \\ \psi_L = \frac{1}{2} (1 - \gamma_5) \psi = \begin{pmatrix} 0 \\ b \end{pmatrix} \end{array} \right\} \text{CHIRAL} \quad \text{IN WEYL REP.}$$

$$\Rightarrow \bar{\psi} (i \gamma \cdot \partial - m) \psi$$

$$= \bar{\psi}_R i \gamma \cdot \partial \psi_R + \bar{\psi}_L i \gamma \cdot \partial \psi_L + m (\bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R)$$

$$\psi_R \longleftrightarrow \psi_L \quad \text{UNDER } P$$

$$\psi_R \sim \text{HELICITY STATE IF } M=0$$

SECOND QUANTIZATION: BOSONS

HILBERT SPACE

$$\mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots$$

FOCK SPACE

ISOMORPHISM BETWEEN FOCK SPACE
AND AN ∞ NUMBER OF HARMONIC
OSCILLATORS

$$H_{\text{h.o.}} = \frac{1}{2} (\hat{p}^2 + q^2)$$

$$= \frac{1}{2} \left[\underbrace{(\hat{p} + iq)}_{a^+} \underbrace{(\hat{p} - iq)}_a + 1 \right]$$

$$\begin{cases} [p, p] = [q, q] = 0 \\ [q, p] = i \end{cases}$$

$$\begin{cases} [a, a] = [a^+, a^+] = 0 \\ [a, a^+] = 1 \end{cases}$$

$|0\rangle = \text{FUNDAMENTAL STATE} :$

$$a|0\rangle = 0$$

$$H|0\rangle = \frac{1}{2}|0\rangle$$

$\Rightarrow (a^+)^n |0\rangle$ IS AN EIGENSTATE

$$\begin{aligned} H(a^+)^n |0\rangle &= (a^+)^n (H+n) |0\rangle \\ &= (n + \frac{1}{2})(a^+)^n |0\rangle \end{aligned}$$

SEVERAL INDEPENDENT OSCILLATORS

$$H = \frac{1}{2} \sum_a \hat{p}_a^2 + q_a^2$$

$$|0\rangle = \bigotimes_a |0_a\rangle$$

$$[a_a^+, a_b^+] = [a_a, a_b] = 0$$

$$[a_a, a_b^+] = \delta_{ab}$$

$$a_a |0\rangle = 0$$

NORMALISATION

$$|A\rangle = \sum \alpha_n a_n^+ |0\rangle$$

$$|B\rangle = \sum \beta_n a_n^+ |0\rangle$$

$$\begin{aligned} \langle B|A\rangle &= \sum \beta_m^* \alpha_n \langle 0|a^m a^{+n}|0\rangle \\ &= \sum \beta_m^* \alpha_n \langle 0|a^{m-1} a^{+n} a + a^{m-1} n a^{+n-1}|0\rangle \\ &= \sum \beta_m^* \alpha_n n! \delta_{mn} \end{aligned}$$

FOCK STATES :

FREE STATES $|\alpha^1\rangle \dots |\alpha^n\rangle \quad \langle \alpha^a | \alpha^b \rangle = \delta^{ab}$

g PARTICLES $|\psi\rangle = \frac{1}{u!} \sum_{perm} |\alpha_1^a\rangle |\alpha_2^b\rangle \dots |\alpha_u^g\rangle$

$$\langle \psi | \psi \rangle = 1 \quad \text{IF STATES ARE DIFFERENT}$$

$$= n! \quad \text{IF } n \text{ STATES ARE IDENTICAL}$$

$$\frac{1}{\sqrt{2}} (|\alpha^1 \alpha^1\rangle + |\alpha^1\rangle |\alpha^1\rangle) = \sqrt{2} |\alpha^1\rangle |\alpha^1\rangle \rightarrow \text{NORM 2}$$

$$= n_1! n_2! \dots n_n!$$

$$\boxed{N = \sum a_i^+ a_i} = \text{TOTAL NUMBER}$$

$$a_\beta |\dots n_\beta \dots\rangle = \sqrt{n_\beta} |\dots, n_{\beta-1}, \dots\rangle$$

$$a_\beta^+ |\dots n_\beta \dots\rangle = \sqrt{n_\beta+1} |\dots, n_{\beta+1}, \dots\rangle$$

$$|\alpha^a\rangle$$

$$\sqrt{n!} (|\alpha^a\rangle |\alpha^a\rangle \dots |\alpha^a\rangle)$$

- 3 BOSONS IN $|\alpha^3\rangle$
- 1 BOSON IN $|\alpha^2\rangle$
- 2 BOSONS IN $|\alpha^1\rangle$

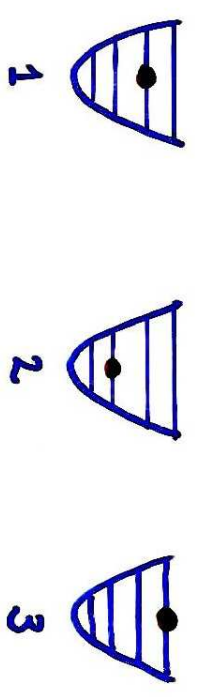
INTERACTION:

$$V(\vec{z}) \quad \langle \alpha^i | V | \alpha^j \rangle = V^{ij}$$

$$V(\vec{z}_i - \vec{z}_j) \quad \langle \alpha^i \alpha^k | V | \alpha^l \alpha^m \rangle = V^{iklm}$$

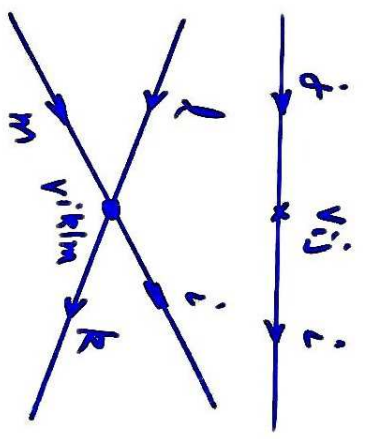
$$a_a^\dagger |0\rangle$$

$$(a_a^\dagger)^n |0\rangle$$



$$V = \sum V^{ij} a_i^\dagger a_j$$

$$V = \sum V^{iklm} a_i^\dagger a_k^\dagger a_l a_m$$



LORENTZ TRANSFORMATIONS

$$x' = \Lambda x + a$$

$$|\phi'\rangle = U(\Lambda, a) |\phi\rangle$$

↑

IRREDUCIBLE REPRESENTATION OF $SO(3,1)$

MASS m , SPIN 0

$$\langle x | U(\Lambda, a) |\phi\rangle = \langle x | \phi'\rangle = \langle \Lambda^{-1}(x-a) | \phi\rangle$$

$$\langle k | \phi'\rangle = e^{ik \cdot a} \langle \Lambda^{-1}k | \phi\rangle$$

N-PARTICLE STATES :

$$\langle k_1, \dots, k_n | U(\Lambda, a) = e^{i \sum k_i \cdot a} \langle \Lambda^{-1}k_1, \dots, \Lambda^{-1}k_n |$$

$$\langle k_1, \dots, k_n | U(\Lambda, a) a_k | \phi\rangle$$

$$= \langle \Lambda^{-1}k_1, \dots, \Lambda^{-1}k_n | a_k | \phi\rangle e^{i \sum k_i \cdot a}$$

$$= \sqrt{n+1} \langle k, \Lambda^{-1}k_1, \dots, \Lambda^{-1}k_n | \phi\rangle e^{i \sum k_i \cdot a}$$

$$= \langle k_1, \dots, k_n | e^{-i \Lambda k \cdot a} a(\Lambda k) U(\Lambda, a) | \phi\rangle$$

$$= \langle k_1, \dots, k_n | U(\Lambda, a) U^{-1}(\Lambda, a) a(\Lambda k) U(\Lambda, a) | \phi\rangle e^{-i \Lambda k \cdot a}$$

$$\Rightarrow \begin{cases} U(\Lambda, a) a_k U(\Lambda, a)^{-1} = e^{-i \Lambda k \cdot a} a(\Lambda k) \\ U(\Lambda, a) a_k^+ U(\Lambda, a)^{-1} = e^{i \Lambda k \cdot a} a^+(\Lambda k) \end{cases}$$

FIELDS

CAN WE DEFINE OPERATORS THAT TRANSFORM LIKE SCALARS, I.E.

$$U \phi(x) U^{-1} = \phi(\Lambda x + a) \quad ?$$

$$\phi^{(+)}(x) = \int_{k_0 > 0} \frac{d^3 k}{k_0} \frac{1}{\sqrt{2(2\pi)^3}} e^{-ik \cdot x} a_{\vec{k}}$$

$$(\square + m^2) \phi^{(+)}(x) = 0 \quad k_0 = \sqrt{\vec{k}^2 + m^2}$$

$$\phi^{-}(x) = \int_{k_0 > 0} \frac{d^3 k}{k_0} \frac{1}{\sqrt{2(2\pi)^3}} e^{+ik \cdot x} a_{\vec{k}}^{\dagger}$$

$$U(\Lambda, a) \phi^{+}(x) U(\Lambda, a)^{-1} = \int_{+} d\Omega(k) e^{-ik \cdot x} U a_{\vec{k}} U^{-1}$$

$$= \int_{+} d\Omega(k) e^{-i\Lambda k \cdot \Lambda x} e^{-i\Lambda k \cdot a} a_{\Lambda \vec{k}}$$

$$= \int_{+} d\Omega(k') e^{-ik' \cdot (\Lambda x + a)} a_{\vec{k}'}$$

$$= \phi^{+}(\Lambda x + a)$$

SIMILARLY FOR ϕ^{-}

HERMITIAN FIELD

$$\phi(x) = \phi^+(x) + \phi^-(x)$$

$$(\square + m^2)\phi(x) = 0$$

$$[\phi(x), \phi(x')] = \frac{i}{2(2\pi)^3} \int_{k_0 > 0} \frac{d^3k}{k_0} \left(e^{-ik(x-x')} - e^{ik(x-x')} \right)$$

prove this \rightarrow

$$\begin{cases} = \frac{i}{(2\pi)^3} \int_{k_0 > 0} e^{ik(x-x')} \frac{d^3k}{k_0} \sin k_0(x_0 - x'_0) \\ = 0 \quad \text{IF} \quad x_0 = x'_0 \end{cases}$$

$$\equiv \Delta(x - x')$$

$$\Delta(\vec{x}, t) = \frac{i}{4\pi} \frac{1}{z} \partial_z F(z, t)$$

$$F(z, t) = \begin{cases} J_0(m\sqrt{t^2 - z^2}) & t > z \\ 0 & -z < t < z \\ -J_0(m\sqrt{t^2 - z^2}) & t < -z \end{cases}$$

$$\Delta(x) = -\frac{i}{2\pi} \varepsilon(t) \left\{ \delta(x^2) - \frac{m^2}{2} \theta(x^2) \frac{J_2(m\sqrt{x^2})}{m\sqrt{x^2}} \right\}$$

CONJUGATE FIELD

$$\pi(x) = \partial_t \phi(x)$$

$$= -\frac{i}{\sqrt{2(2\pi)^3}} \int_{k_0 > 0} d^3k (e^{-ik \cdot x} a_{\vec{k}} - e^{ik \cdot x} a_{\vec{k}}^\dagger)$$

$$\boxed{[\pi(x), \phi(x')]_{t=t'} = -i \delta^{(3)}(\vec{x} - \vec{x}')} \quad \text{prove this}$$

\Rightarrow WE RECOVER THE
LAGRANGIAN COMMUTATORS!

RELATIVISTIC NORMALISATION

$$|\vec{p}\rangle = c a_{\vec{p}}^{\dagger} |0\rangle$$

$$\langle \vec{p} | \vec{q} \rangle = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) c^2$$

$$\hat{\rightarrow} |\vec{p}'\rangle$$

$$\langle \vec{p}' | \vec{q}' \rangle = (2\pi)^3 \delta^{(3)}(\vec{p}' - \vec{q}') c^2$$

$$\int d^3 p' \delta^{(3)}(\vec{p}' - \vec{q}') = \int \frac{p'_0}{p_0} \delta^{(3)}(\vec{p}' - \vec{q}') d^3 p$$

$\frac{d^3 p}{p_0}$ invariant \rightarrow replace $d^3 p$ with this

$$\Rightarrow p_0 \delta^{(3)}(\vec{p} - \vec{q}) = p'_0 \delta^{(3)}(\vec{p}' - \vec{q}')$$

ABSORB THIS IN THE NORMALISATION

$$|\vec{p}\rangle = \sqrt{2E_p} a_{\vec{p}}^{\dagger} |0\rangle \quad [|\vec{p}\rangle = \text{GeV}^{-1}]$$

$$\langle \vec{p} | \vec{q} \rangle = (2\pi)^3 2E_p \delta^{(3)}(\vec{p} - \vec{q}) \quad [\langle \vec{p} | \vec{q} \rangle] = \text{GeV}^{-2}$$

$$[a_{\vec{p}}, a_{\vec{p}'}^{\dagger}] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}') \quad [a] = \text{GeV}^{-3/2}$$

$$\phi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^{\dagger} e^{-i\vec{p}\cdot\vec{x}})$$

$$[\phi] = \text{GeV}$$

RECIPE

$$S = \int d^4x \mathcal{L}$$

• SYMMETRIES LEAVE S INVARIANT

• S EXTREMUM

→ EULER-LAGRANGE FOR FIELDS

• CONJUGATE MOMENTA π

EQUAL
TIMES

$$\begin{cases} [\pi, \phi] = i\delta^{(3)} \\ [\pi, \pi] = [\phi, \phi] = 0 \end{cases}$$

• EXPAND IN TERMS OF EIGEN FUNCTIONS

→ COEFFICIENTS ARE CREATION + ANNIHILATION OPERATORS

→ HILBERT SPACE KNOWN

• EVOLVE OPERATORS USING H

$$i\partial_t O = [H, O]$$

• ONLY ONE FIELD IN ALL SPACETIME

→ PARTICLES ARE IDENTICAL