1. CLASSICAL ELECTRODYNAMICS

- MAXWELL'S EQUATIONS
- LORENTZ GROUP
- ANTIPARTICLES

MAXWELL'S EQUATIONS

$$\begin{cases} \varepsilon_0 \vec{\nabla}\cdot\vec{E}_{SI} = \rho_{SI} \\ \frac{1}{\mu_0} \vec{\nabla}\times\vec{B}_{SI} - \varepsilon_0 \partial_t \vec{E}_{SI} = \vec{J}_{SI} \\ \vec{\nabla}\times\vec{E}_{SI} + \partial_t \vec{B}_{SI} = 0 \\ \vec{\nabla}\cdot\vec{B}_{SI} = 0 \end{cases}$$

$$\partial_t \equiv \partial_0 \equiv \frac{\partial}{\partial t}$$

→ HEAVYSIDE - LORENTZ

$$q = \frac{q_{SI}}{\sqrt{\varepsilon_0}} \qquad \vec{E} = \sqrt{\varepsilon_0}\,\vec{E}_{SI} \qquad \vec{B} = \frac{\vec{B}_{SI}}{\sqrt{\mu_0}}$$

$$\begin{cases} \vec{\nabla}\cdot\vec{E} = \rho \\ \vec{\nabla}\times\vec{B} - \frac{1}{c}\partial_t\vec{E} = \frac{1}{c}\vec{J} \\ \vec{\nabla}\times\vec{E} + \frac{1}{c}\partial_t\vec{B} = 0 \\ \vec{\nabla}\cdot\vec{B} = 0 \end{cases}$$

$$\vec{F}_c = \frac{1}{4\pi}\frac{qq'}{r^2} \qquad \vec{F}_L = q\left(\vec{E} + \frac{\vec{v}}{c}\vec{B}\right)$$

$$[q] = M^{1/2} L^{3/2} T^{-1}$$ "stat coulomb"

→ NATURAL UNITS

$\begin{cases} v \text{ in units of } c \\ \vec{L} \text{ in units of } \hbar \end{cases}$

$$\boxed{\begin{cases} \vec{\nabla}\cdot\vec{E} = \rho \\ \vec{\nabla}\times\vec{B} - \partial_t\vec{E} = \vec{J} \\ \vec{\nabla}\times\vec{E} + \partial_t\vec{B} = 0 \\ \vec{\nabla}\cdot\vec{B} = 0 \end{cases}}$$

how much is
1 Tesla
in natural units?

$$[q] = 1 \quad \hbar^{1/2} c^{1/2}$$

SIMPLIFICATION

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \Leftrightarrow \quad \vec{B} = \vec{\nabla}\times\vec{A}$$

$$\vec{\nabla}\times(\vec{E}+\partial_t\vec{A}) = 0 \quad \Leftrightarrow \quad \vec{E} = -\vec{\nabla}\phi - \partial_t\vec{A}$$

NOTATION

$$x^\mu = (t, x_1, x_2, x_3) \qquad \partial_\mu = \frac{\partial}{\partial x^\mu}$$

$$x_\mu = (t, -x_1, -x_2, -x_3) \qquad \partial^\mu = \frac{\partial}{\partial x_\mu}$$

$$J^\mu = (\rho, J_1, J_2, J_3)$$

$$A^\mu = (\phi, A_1, A_2, A_3)$$

$$x\cdot y = \sum_\mu x^\mu y_\mu$$

CHECK

$$\boxed{F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu} = \begin{pmatrix} \cdot & -E_1 & -E_2 & -E_3 \\ E_1 & \cdot & -B_3 & B_2 \\ E_2 & B_3 & \cdot & -B_1 \\ E_3 & -B_2 & B_1 & \cdot \end{pmatrix}$$

FIELD STRENGTH TENSOR

→ INHOMOGENEOUS EQUATIONS :

$$\boxed{\partial_\mu F^{\mu\nu} = J^\nu}$$

HOMOGENEOUS : $\tilde{F}_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$

$$\partial_\mu \tilde{F}_{\mu\nu} = 0$$

THE VECTOR POTENTIAL A^μ

$$\boxed{\Box A^\nu - \partial^\nu(\partial^\mu A_\mu) = J^\nu}$$

⇒ • CURRENT CONSERVATION

$$\boxed{\partial_\nu J^\nu = 0}$$

• GAUGE INVARIANCE

$$\boxed{A^\nu \longrightarrow A^\nu + \partial^\nu \Lambda}$$

GAUGE CHOICE : GAUGE CONDITION

$\partial_\mu A^\mu$	$= 0$	LORENZ
$n_\mu A^\mu$	$= 0$	AXIAL
$\vec{\nabla}\cdot\vec{A}$	$= 0$	COULOMB

WHAT IS THE FIELD OF A STATIC CHARGE IN EACH GAUGE ? WHAT IS THE CORRESPONDING A^μ ?

$$\boxed{\begin{cases} \Box A^\nu = J^\nu \\ \partial_\mu A^\mu = 0 \end{cases}}$$

INVARIANCE PROPERTIES:

WHAT ARE THE TRANSFORMATIONS

$$(t, \vec{x}) \to (t', \vec{x}')$$

THAT LEAVE THE EQUATIONS INVARIANT?

2-d :
$$\begin{cases} \Box = \partial_t^2 - \partial_x^2 \\ \partial_\mu A^\mu = \partial_t A^0 + \partial_x A^1 \end{cases}$$

- WORK IT OUT FOR $\Delta = \partial_x^2 + \partial_y^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} = M \begin{pmatrix} x' \\ y' \end{pmatrix} \qquad \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = M^{-1} \begin{pmatrix} \partial_{x'} \\ \partial_{y'} \end{pmatrix}$$

$$\Delta' = (\partial_{x'}, \partial_{y'}) \begin{pmatrix} \partial_{x'} \\ \partial_{y'} \end{pmatrix} = (\partial_x, \partial_y)\, M^T M \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}$$

$$= \Delta \quad \text{if} \quad M^T M = \mathbb{1}$$

ORTHOGONAL GROUP SO(2)

$$M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

- $t = i\xi \quad \Rightarrow \quad \Box = -(\partial_\xi^2 + \partial_x^2)$

$$\begin{cases} x' = \cos\theta\, x + \sin\theta\, \xi \\ \xi' = -\sin\theta\, x + \cos\theta\, \xi \end{cases}$$

$$\theta = i\varphi \qquad \sin\theta = i \sinh\varphi \qquad \cos\theta = \cosh\varphi$$

$$\begin{cases} x' = \cosh\varphi\, x + \sinh\varphi\, t \\ t' = \sinh\varphi\, x + \cosh\varphi\, t \end{cases} \qquad \underline{SO(1,1)}$$

leaves $\Box$ invariant

INTERPRETATION

$$x' = 0 \quad \Rightarrow \quad x = -\underbrace{\tanh\varphi}_{v}\, t$$

$$\Rightarrow \quad -1 < v < 1$$

$$\left.\begin{aligned} \cosh\varphi &= \frac{1}{\sqrt{1-v^2}} \\ \sinh\varphi &= \frac{v}{\sqrt{1-v^2}} \end{aligned}\right\} \Rightarrow \begin{cases} x' = \frac{1}{\sqrt{1-v^2}}(x + vt) \\ t' = \frac{1}{\sqrt{1-v^2}}(t + vx) \end{cases}$$

ALSO LEAVE $x^\mu x_\mu$ $\quad \partial^\mu x_\mu$ INVARIANT

$\Rightarrow$ MAXWELL'S EQUATIONS ARE INVARIANT UNDER ROTATIONS + BOOSTS

= PROPER LORENTZ GROUP

IF $x^\mu, \partial^\mu, A^\mu$ TRANSFORM IN THE SAME WAY

NOTE

Show that φ is additive and that $\varphi \sim v$ if $c \to \infty$

$$p^\mu = (E, \vec{p})$$

$$\tanh\varphi = v \quad \Rightarrow \quad \varphi = \frac{1}{2}\log\frac{1+\beta}{1-\beta}$$

$$= \frac{1}{2}\log\frac{p+E}{p-E}$$

RAPIDITY

LORENTZ GROUP

PROPER — ROTATIONS + BOOSTS

IMPROPER — PARITY : $\vec{x} \to -\vec{x}$, $t \to t$

TIME REVERSAL : $\vec{x} \to \vec{x}$, $t \to -t$

PROPER GROUP : $SO(3,1)$

$\longrightarrow$ REPRESENTATIONS

- SCALAR $\Lambda_P S = S$ $\Lambda_I S = S$
- PSEUDOSCALAR $\Lambda_P P = P$ $\Lambda_I P = -P$
- VECTOR x^μ
- PSEUDOVECTOR p^μ $\Lambda_I p = p$
- TENSOR $F^{\mu\nu}$

\+ SPINORS

CPT

LOCALITY
CAUSALITY
UNITARITY
LORENTZ PROPER

$\Rightarrow$ $\bar{q}$ $t<0$ $\Leftrightarrow$ q $t>0$

antiparticle at $(-\vec{x},-t)$ particle at (x,t)

C
P
T
not conserved

CP
PT
CT
not conserved

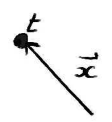
by weak interactions

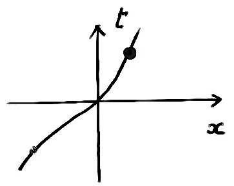
CLASSICAL ANTIPARTICLES

WE NEED A MODEL FOR J^μ

$$J^\mu \div q \frac{dx^\mu}{d\tau}$$

τ MUST BE AN INVARIANT





$$(d\tau)^2 = (dt)^2 - (d\vec{x})^2$$

$$\tau = \int d\tau = \pm \int_0^t dt \sqrt{1-\left(\frac{dx}{dt}\right)^2}$$

PROPER TIME

$\tau > 0$ NORMAL PARTICLE

$\tau < 0$ $\left\{ \begin{array}{l} \text{SAME } x(t) \\ \text{OPPOSITE CHARGE} \end{array} \right.$

$$q\left(-\sqrt{1-v^2}\right) = (-q)\sqrt{1-v^2}$$

GRAVITY INVOLVES $T_{\mu\nu} \sim m v^2$

$\rightarrow$ $(d\tau)^2$ UNCHANGED

$\rightarrow$ m UNCHANGED

2. THE PRINCIPLE OF EXTREMAL ACTION

- LAGRANGIAN
- HAMILTONIAN
- POISSON BRACKETS
- SYMETRIES
- LAGRANGIAN OF QED

CLASSICAL LAGRANGIAN

$$S = \int_{t_1}^{t_2} L(q_i(t), \dot{q}_i(t))\, dt$$

$$\frac{\delta S}{\delta Q_i} = 0 \quad \Rightarrow \quad Q_i(t) \text{ ARE THE PHYSICAL TRAJECTORIES}$$

$$q_i(t) = Q_i(t) + \delta q_i(t)$$

$$S(q) = S(Q) + \int_{t_1}^{t_2} \left.\frac{\delta L}{\delta q_i(t)}\right|_{Q(t)} \delta q_i(t)$$

ASSUME
$$\begin{cases} \delta q(t_1) = \delta q(t_2) = 0 \\ \delta \frac{d}{dt} q(t) = \frac{d}{dt}\, \delta q(t) \end{cases}$$

$$\delta S = \sum_i \int_{t_1}^{t_2} \frac{\partial L}{\partial q_i}\, \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt}\, \delta q_i$$

$$= \sum_i \int_{t_1}^{t_2} \delta q_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) = 0$$

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0$$

CLASSICAL HAMILTONIAN

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$$

INVERT $\Rightarrow$ $\dot{q}_i(q_j, p_j)$

$$H(p,q) = \sum_j p_j\, \dot{q}_j(q,p) - L(q, \dot{q}(q,p))$$

$$\Rightarrow \begin{cases} \dot{q}_i = \frac{\partial H}{\partial p_i} \\ \dot{p}_i = -\frac{\partial H}{\partial q_i} \end{cases}$$ Prove this

$$f(p_i, q_i)$$

$$\frac{df}{dt} = \sum_i \frac{\partial f}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q_i}$$

$$= -\{f, H\}$$

$$\{a, b\} \equiv \sum_i \frac{\partial a}{\partial p_i} \frac{\partial b}{\partial q_i} - \frac{\partial a}{\partial q_i} \frac{\partial b}{\partial p_i}$$

Find, for a single particle, the action which is invariant w.r.t. boosts. Obtain the Euler-Lagrange equation, the momentum and the energy.

FIELDS :

$$q_i \longrightarrow q(y)$$

$$L \longrightarrow \int d^3y\, \mathcal{L}(q, \underbrace{\dot{q}, \vec{\nabla}_y q}_{\partial_\mu q})$$

$$S = \int d^4y\, \mathcal{L}(q, \partial_\mu q)$$

$$\delta S = 0$$

$$\Rightarrow \boxed{\partial_\mu \frac{\partial}{\partial_\mu q} \mathcal{L} - \frac{\partial \mathcal{L}}{\partial q} = 0}$$

y = space-time coordinate x

$q = \varphi(y)$

HAMILTONIAN

$$\pi(x) \equiv \int d^3y \frac{\partial \mathcal{L}(\varphi(y), \partial_\mu \varphi)}{\partial \dot{\varphi}(x)}$$

$$H \equiv \int d^3x\, (\pi \dot{\varphi} - \mathcal{L})$$

$$\Rightarrow \begin{cases} \dfrac{\partial H}{\partial \pi} = \dot{\varphi} \\ -\dfrac{\partial H}{\partial \varphi} = \dot{\pi} \end{cases}$$

show this

POISSON BRACKETS

$$\{A, B\} \equiv \frac{\partial A}{\partial \pi}\frac{\partial B}{\partial \varphi} - \frac{\partial A}{\partial \varphi}\frac{\partial B}{\partial \pi}$$

$$\Rightarrow \begin{cases} \dot{\varphi} = \{H, \varphi\} \\ \dot{\pi} = \{H, \pi\} \end{cases}$$

$$\left[\begin{array}{l} \{\pi(x,t), \varphi(y,t)\} = \delta^{(3)}(\vec{x} - \vec{y}) \\ \{\pi, \pi\} = \{\varphi, \varphi\} = 0 \end{array} \right.$$

SIMPLEST EXAMPLE

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi\, \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2$$

Euler - Lagrange	$(\Box + m^2)\varphi = 0$
Conjugate momenta	$\pi = \dot{\varphi}$
Hamiltonian	$H = \frac{1}{2}\int d^3x \left(\pi^2 + (\vec{\nabla}\varphi)^2 + m^2\varphi^2\right)$

NOETHER's THEOREM

CONTINUOUS SYMETRY $\varphi' = U(\alpha)\varphi$

WITH $S(U(\alpha)\varphi) = S(\varphi)$

$\Rightarrow$ CONSERVED CURRENT j^μ

$$\begin{cases} \partial_\mu j^\mu = 0 \\ \frac{d}{dt}\int d^3x\, j_0 = 0 \end{cases}$$

proof:

$$\varphi \to \varphi' = \varphi + \alpha\Delta\varphi$$

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \alpha\,\partial_\mu J^\mu$$

$$\Delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi}(\alpha\Delta\varphi) + \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\,\partial_\mu(\alpha\Delta\varphi)$$

$$= \alpha\Delta\varphi\left(\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\right) + \alpha\,\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\Delta\varphi\right)$$

$$\Rightarrow \boxed{j^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\Delta\varphi - J^\mu}$$

$Q = \int d^3x\, j_0(x)$ CHARGE

EXAMPLES:

- $\mathcal{L} = \partial_\mu\varphi\,\partial^\mu\varphi^* - m^2\varphi\varphi^*$

$\varphi \to e^{i\alpha}\varphi$

$$\begin{cases} \alpha\Delta\varphi = i\alpha\varphi \\ \alpha\Delta\varphi^* = -i\alpha\varphi^* \end{cases}$$

$$J^\mu = 0$$

$$j^\mu = i\left[(\partial^\mu\varphi^*)\varphi - \varphi^*(\partial^\mu\varphi)\right]$$

- $x_\lambda \to x_\lambda + a_\lambda$ POINCARE

$$\varphi \to \varphi + a_\lambda(\partial^\lambda\varphi)\Delta\varphi$$

$$\mathcal{L} \to \mathcal{L} + a_\nu\partial^\nu\mathcal{L} = \mathcal{L} + a_\lambda\,\partial^\mu\underbrace{(\delta^\lambda_\mu\mathcal{L})}_{J_\mu}$$

$$\Rightarrow j^\mu = \frac{\partial\mathcal{L}\Delta\varphi}{\partial\partial_\mu\varphi} - J^\mu$$

$$= \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\partial_\lambda\varphi - \mathcal{L}\,\delta^\mu_\lambda \equiv T^\mu{}_\lambda$$

ENERGY-MOMENTUM TENSOR

$$\begin{cases} H = \int T^{00}\,d^3x = \int \mathcal{H}\,d^3x \\ \vec{P} = \int T^{0i}\,d^3x \end{cases}$$

LAGRANGIAN FOR MAXWELL

GAUGE INVARIANCE $\Rightarrow$ $F_{\mu\nu}$

LORENTZ INVARIANCE

$$S = \int d^4x \, \mathcal{L}$$

$$= \int d^4x \boxed{\left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu \right)}$$

$$\Rightarrow \quad \begin{cases} \partial_\mu F^{\mu\nu} = j^\nu \\ \partial_\nu j^\nu = 0 \end{cases}$$

NOETHER

$A_\mu \to A_\mu + \partial_\mu \Lambda$

<u>NOTE:</u> ONE COULD HAVE ADDED

$$\tilde{F}_{\mu\nu} F^{\mu\nu} = \varepsilon_{\alpha\beta\mu\nu} F^{\alpha\beta} F^{\mu\nu}$$

SHOW THAT $F_{\mu\nu} \tilde{F}^{\mu\nu}$ IS A TOTAL DERIVATIVE

$$\pi^\rho = \frac{\partial \mathcal{L}}{\partial \dot{A}_\rho} = F^{\rho 0} \qquad \Rightarrow \pi^0 = 0$$

⚠ $\rightarrow$ PROBLEM WITH $\{A^0, \pi^0\}$!