1. CLASSICAL ELECTRODYNAMICS

- MAXWELL'S EQUATIONS
- LORENTZ GROUP
- ANTIPARTICLES

MAXWELL'S EQUATIONS

$$\begin{cases} \varepsilon_0 \vec{\nabla}\cdot\vec{E}_{SI} = \rho_{SI} \\ \frac{1}{\mu_0}\vec{\nabla}\times\vec{B}_{SI} - \varepsilon_0 \partial_t \vec{E}_{SI} = \vec{J}_{SI} \\ \vec{\nabla}\times\vec{E}_{SI} + \partial_t \vec{B}_{SI} = 0 \\ \vec{\nabla}\cdot\vec{B}_{SI} = 0 \end{cases}$$

$$\partial_t \equiv \partial_0 \equiv \frac{\partial}{\partial t}$$

$$q = \frac{q_{SI}}{\sqrt{\varepsilon_0}} \qquad \vec{E} = \sqrt{\varepsilon_0}\,\vec{E}_{SI} \qquad \vec{B} = \frac{\vec{B}_{SI}}{\sqrt{\mu_0}}$$

→ HEAVYSIDE - LORENTZ

$$\begin{cases} \vec{\nabla}\cdot\vec{E} = \rho \\ \vec{\nabla}\times\vec{B} - \frac{1}{c}\partial_t\vec{E} = \frac{1}{c}\vec{J} \\ \vec{\nabla}\times\vec{E} + \frac{1}{c}\partial_t\vec{B} = 0 \\ \vec{\nabla}\cdot\vec{B} = 0 \end{cases}$$

$$\vec{F}_c = \frac{1}{4\pi}\frac{qq'}{r^2} \qquad \vec{F}_L = q\left(\vec{E} + \frac{\vec{v}}{c}\vec{B}\right)$$

$$[q] = M^{1/2} L^{3/2} T^{-1}$$ "stat coulomb"

→ NATURAL UNITS

v in units of c
$\vec{L}$ in units of $\hbar$

$$\boxed{\begin{cases} \vec{\nabla}\cdot\vec{E} = \rho \\ \vec{\nabla}\times\vec{B} - \partial_t\vec{E} = \vec{J} \\ \vec{\nabla}\times\vec{E} + \partial_t\vec{B} = 0 \\ \vec{\nabla}\cdot\vec{B} = 0 \end{cases}}$$

how much is
1 Tesla
in natural units?

$$[q] = 1 \quad \hbar^{1/2} c^{1/2}$$

SIMPLIFICATION

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \Leftrightarrow \quad \vec{B} = \vec{\nabla}\times\vec{A}$$

$$\vec{\nabla}\times(\vec{E}+\partial_t\vec{A}) = 0 \quad \Leftrightarrow \quad \vec{E} = -\vec{\nabla}\phi - \partial_t\vec{A}$$

NOTATION

$$x^\mu = (t, x_1, x_2, x_3) \qquad \partial_\mu = \frac{\partial}{\partial x^\mu}$$

$$x_\mu = (t, -x_1, -x_2, -x_3) \qquad \partial^\mu = \frac{\partial}{\partial x_\mu}$$

$$J^\mu = (\rho, J_1, J_2, J_3)$$

$$A^\mu = (\phi, A_1, A_2, A_3)$$

$$x\cdot y = \sum_\mu x^\mu y_\mu$$

CHECK

$$\boxed{F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu} = \begin{pmatrix} \cdot & -E_1 & -E_2 & -E_3 \\ E_1 & \cdot & -B_3 & B_2 \\ E_2 & B_3 & \cdot & -B_1 \\ E_3 & -B_2 & B_1 & \cdot \end{pmatrix}$$

FIELD STRENGTH TENSOR

→ INHOMOGENEOUS EQUATIONS:

$$\boxed{\partial_\mu F^{\mu\nu} = J^\nu}$$

HOMOGENEOUS :

$$\tilde{F}_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$$

$$\partial_\mu \tilde{F}_{\mu\nu} = 0$$

THE VECTOR POTENTIAL A^μ

$$\boxed{\Box A^\nu - \partial^\nu(\partial^\mu A_\mu) = J^\nu}$$

$\Rightarrow$

- CURRENT CONSERVATION

$$\boxed{\partial_\nu J^\nu = 0}$$

- GAUGE INVARIANCE

$$\boxed{A^\nu \longrightarrow A^\nu + \partial^\nu \Lambda}$$

GAUGE CHOICE :	GAUGE CONDITION
$\partial_\mu A^\mu = 0$	LORENZ
$n_\mu A^\mu = 0$	AXIAL
$\vec{\nabla} \cdot \vec{A} = 0$	COULOMB

WHAT IS THE FIELD OF A STATIC CHARGE IN EACH GAUGE ? WHAT IS THE CORRESPONDING A^μ ?

$$\boxed{\begin{cases} \Box A^\nu = J^\nu \\ \partial_\mu A^\mu = 0 \end{cases}}$$

INVARIANCE PROPERTIES:

WHAT ARE THE TRANSFORMATIONS

$$(t, \vec{x}) \to (t', \vec{x}')$$

THAT LEAVE THE EQUATIONS INVARIANT ?

2-d :
$$\begin{cases} \Box = \partial_t^2 - \partial_x^2 \\ \partial_\mu A^\mu = \partial_t A^0 + \partial_x A^1 \end{cases}$$

- WORK IT OUT FOR $\Delta = \partial_x^2 + \partial_y^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} = M \begin{pmatrix} x' \\ y' \end{pmatrix} \qquad \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = M^{-1} \begin{pmatrix} \partial_{x'} \\ \partial_{y'} \end{pmatrix}$$

$$\Delta' = (\partial_{x'}, \partial_{y'}) \begin{pmatrix} \partial_{x'} \\ \partial_{y'} \end{pmatrix} = (\partial_x, \partial_y)\, M^T M \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}$$

$$= \Delta \quad \text{if} \quad M^T M = \mathbb{1}$$

ORTHOGONAL GROUP $SO(2)$

$$M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

- $t = i\xi \quad \Rightarrow \quad \Box = -\left(\partial_\xi^2 + \partial_x^2\right)$

$$\begin{cases} x' = \cos\theta\, x + \sin\theta\, \xi \\ \xi' = -\sin\theta\, x + \cos\theta\, \xi \end{cases}$$

$$\theta = i\varphi \qquad \sin\theta = i \sinh\varphi \qquad \cos\theta = \cosh\varphi$$

$$\begin{cases} x' = \cosh\varphi\; x + \sinh\varphi\; t \\ t' = \sinh\varphi\; x + \cosh\varphi\; t \end{cases} \qquad SO(1,1)$$

leaves $\Box$ invariant

INTERPRETATION

$$x' = 0 \quad \Rightarrow \quad x = -\underbrace{\tanh\varphi}_{v}\, t$$

$$\Rightarrow \quad -1 < v < 1$$

$$\left.\begin{aligned} \cosh\varphi &= \frac{1}{\sqrt{1-v^2}} \\ \sinh\varphi &= \frac{v}{\sqrt{1-v^2}} \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} x' &= \frac{1}{\sqrt{1-v^2}}(x+vt) \\ t' &= \frac{1}{\sqrt{1-v^2}}(t+vx) \end{aligned}\right.$$

ALSO LEAVE $x^\mu x_\mu$ $\partial^\mu x_\mu$ INVARIANT

$\Rightarrow$ MAXWELL'S EQUATIONS ARE INVARIANT UNDER ROTATIONS + BOOSTS

$\equiv$ PROPER LORENTZ GROUP

IF $x^\mu, \partial^\mu, A^\mu$ TRANSFORM IN THE SAME WAY

NOTE

$$p^\mu = (E, \vec{p})$$

Show that y is additive and that $y \sim v$ if $c \to \infty$

$$\tanh\varphi = v \quad \Rightarrow \quad \varphi = \frac{1}{2}\log\frac{1+\beta}{1-\beta}$$

$$= \frac{1}{2}\log\frac{p+E}{p-E}$$

RAPIDITY

LORENTZ GROUP

PROPER — ROTATIONS + BOOSTS

IMPROPER — PARITY : $\vec{x} \to -\vec{x}$

$t \to t$

TIME REVERSAL : $\vec{x} \to \vec{x}$

$t \to -t$

PROPER GROUP : $SO(3,1)$

$\longrightarrow$ REPRESENTATIONS

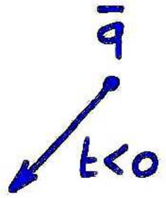
- SCALAR $\Lambda_P S = S$ $\Lambda_I S = S$
- PSEUDOSCALAR $\Lambda_P P = P$ $\Lambda_I P = -P$
- VECTOR x^μ
- PSEUDOVECTOR p^μ $\Lambda_I p = p$
- TENSOR $F^{\mu\nu}$

+ SPINORS

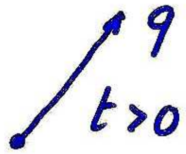
CPT

LOCALITY
CAUSALITY
UNITARITY
LORENTZ PROPER

$\Rightarrow$



$\Leftrightarrow$



antiparticle at $(-\vec{x}, -t)$

particle at (x, t)

C, P, T not conserved

CP, PT, CT not conserved

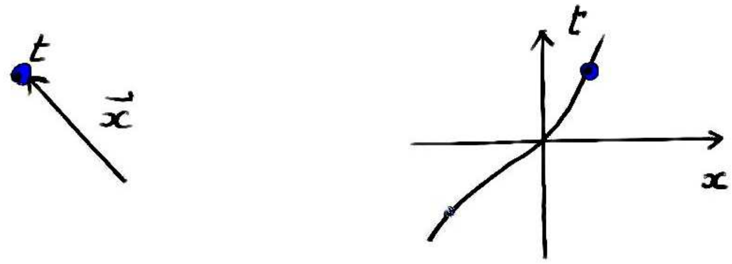
by weak interactions

CLASSICAL ANTIPARTICLES

WE NEED A MODEL FOR J^μ

$$J^\mu \doteq q \frac{dx^\mu}{d\tau}$$

τ MUST BE AN INVARIANT



$$(d\tau)^2 = (dt)^2 - (d\vec{x})^2$$

$$\tau = \int d\tau = \pm \int_0^t dt \sqrt{1-\left(\frac{dx}{dt}\right)^2}$$

PROPER TIME

$\tau > 0$ NORMAL PARTICLE

$\tau < 0$ $\begin{cases} \text{SAME } x(t) \\ \text{OPPOSITE CHARGE} \end{cases}$

$$q\left(-\sqrt{1-v^2}\right) = (-q)\sqrt{1-v^2}$$

GRAVITY INVOLVES $T_{\mu\nu} \sim m v^2$

$\rightarrow$ $(d\tau)^2$ UNCHANGED

$\rightarrow$ m UNCHANGED

2. THE PRINCIPLE OF EXTREMAL ACTION

- LAGRANGIAN
- HAMILTONIAN
- POISSON BRACKETS
- SYMETRIES
- LAGRANGIAN OF QED

CLASSICAL LAGRANGIAN

$$S = \int_{t_1}^{t_2} L(q_i(t), \dot{q}_i(t))\, dt$$

$$\frac{\delta S}{\delta Q_i} = 0 \quad \Rightarrow \quad Q_i(t) \text{ ARE THE PHYSICAL TRAJECTORIES}$$

$$q_i(t) = Q_i(t) + \delta q_i(t)$$

$$S(q) = S(Q) + \int_{t_1}^{t_2} \left.\frac{\delta L}{\delta q_i(t)}\right|_{Q(t)} \delta q_i(t)$$

ASSUME

$$\begin{cases} \delta q(t_1) = \delta q(t_2) = 0 \\ \delta \frac{d}{dt} q(t) = \frac{d}{dt} \delta q(t) \end{cases}$$

$$\delta S = \sum_i \int_{t_1}^{t_2} \frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \delta q_i$$

$$= \sum_i \int_{t_1}^{t_2} \delta q_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) = 0$$

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0$$

CLASSICAL HAMILTONIAN

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$$

INVERT $\Rightarrow$ $\dot{q}_i(q_j, p_j)$

$$H(p,q) = \sum_j p_j \dot{q}_j(q,p) - L(q, \dot{q}(q,p))$$

$$\Rightarrow \quad \begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i = -\dfrac{\partial H}{\partial q_i} \end{cases}$$

Prove this

$f(p_i, q_i)$

$$\frac{df}{dt} = \sum_i \frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i}$$

$$= -\{f, H\}$$

$$\{a, b\} \equiv \sum_i \frac{\partial a}{\partial p_i}\frac{\partial b}{\partial q_i} - \frac{\partial a}{\partial q_i}\frac{\partial b}{\partial p_i}$$

Find, for a single particle, the action which is invariant w.r.t. boosts.
Obtain the Euler-Lagrange equation, the momentum and the energy.

FIELDS :

$$q_i \longrightarrow q(y)$$

$$L \longrightarrow \int d^3y \, \mathcal{L}(q, \underbrace{\dot{q}, \vec{\nabla}_y q}_{\partial_\mu q})$$

$$S = \int d^4y \; \mathcal{L}(q, \partial_\mu q)$$

$$\delta S = 0$$

$$\Rightarrow \boxed{\partial_\mu \frac{\partial}{\partial_\mu q} \mathcal{L} - \frac{\partial \mathcal{L}}{\partial q} = 0}$$

y = space-time coordinate x

$q = \varphi(y)$

HAMILTONIAN

$$\pi(x) \equiv \int d^3y \; \frac{\partial \mathcal{L}(\varphi(y), \partial_\mu \varphi)}{\partial \dot{\varphi}(x)}$$

$$H \equiv \int d^3x \; (\pi \dot{\varphi} - \mathcal{L})$$

$$\Rightarrow \begin{cases} \dfrac{\partial H}{\partial \pi} = \dot{\varphi} \\[2ex] -\dfrac{\partial H}{\partial \varphi} = \dot{\pi} \end{cases}$$

show this

POISSON BRACKETS

$$\{A, B\} \equiv \frac{\partial A}{\partial \pi}\frac{\partial B}{\partial \varphi} - \frac{\partial A}{\partial \varphi}\frac{\partial B}{\partial \pi}$$

$$\Rightarrow \begin{cases} \dot{\varphi} = \{H, \varphi\} \\ \dot{\pi} = \{H, \pi\} \end{cases}$$

$$\left[\begin{array}{l} \{\pi(x,t), \varphi(y,t)\} = \delta^{(3)}(\vec{x}-\vec{y}) \\ \{\pi, \pi\} = \{\varphi, \varphi\} = 0 \end{array}\right.$$

SIMPLEST EXAMPLE

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2$$

Euler - Lagrange	$(\Box + m^2)\varphi = 0$
conjugate momenta	$\pi = \dot{\varphi}$
Hamiltonian	$H = \frac{1}{2}\int d^3x\left(\pi^2 + (\vec{\nabla}\varphi)^2 + m^2\varphi^2\right)$

NOETHER's THEOREM

CONTINUOUS SYMETRY $\varphi' = U(\alpha)\varphi$

WITH $S(U(\alpha)\varphi) = S(\varphi)$

$\Rightarrow$ CONSERVED CURRENT j^μ

$$\begin{cases} \partial_\mu j^\mu = 0 \\ \frac{d}{dt}\int d^3x\, j_0 = 0 \end{cases}$$

proof: $\varphi \to \varphi' = \varphi + \alpha\Delta\varphi$

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \alpha\,\partial_\mu J^\mu$$

$$\Delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi}(\alpha\Delta\varphi) + \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\,\partial_\mu(\alpha\Delta\varphi)$$

$$= \alpha\Delta\varphi\left(\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\right) + \alpha\,\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\Delta\varphi\right)$$

$$\Rightarrow \boxed{j^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\Delta\varphi - J^\mu}$$

$Q = \int d^3x\, j_0(x)$ CHARGE

EXAMPLES:

- $\mathcal{L} = \partial_\mu \varphi \, \partial^\mu \varphi^* - m^2 \varphi \varphi^*$

$\varphi \to e^{i\alpha} \varphi$

$$\begin{cases} \alpha \Delta\varphi = i\alpha\varphi \\ \alpha \Delta\varphi^* = -i\alpha\varphi^* \end{cases}$$

$$\mathcal{J}^\mu = 0$$

$$j^\mu = i\left[(\partial^\mu \varphi^*)\varphi - \varphi^*(\partial^\mu \varphi)\right]$$

- $x_\lambda \to x_\lambda + a_\lambda$

POINCARE

$$\varphi \to \varphi + a_\lambda \overbrace{\partial^\lambda \varphi}^{\Delta\varphi}$$

$$\mathcal{L} \to \mathcal{L} + a_\nu \partial^\nu \mathcal{L} = \mathcal{L} + a_\lambda \partial^\mu \underbrace{(\delta^\lambda_\mu \mathcal{L})}_{\mathcal{J}_\mu}$$

$$\Rightarrow \quad j^\mu = \frac{\partial \mathcal{L} \Delta\varphi}{\partial \partial_\mu \varphi} - \mathcal{J}^\mu$$

$$= \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \partial_\lambda \varphi - \mathcal{L} \, \delta^\mu_\lambda \equiv T^\mu{}_\lambda$$

ENERGY - MOMENTUM TENSOR

$$\begin{cases} H = \int T^{00} d^3x = \int \mathcal{H} \, d^3x \\ \vec{P} = \int T^{0i} d^3x \end{cases}$$

LAGRANGIAN FOR MAXWELL

GAUGE INVARIANCE $\Rightarrow$ $F_{\mu\nu}$

LORENTZ INVARIANCE

$$S = \int d^4x \, \mathcal{L}$$

$$= \int d^4x \boxed{\left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu\right)}$$

$$\Rightarrow \quad \begin{cases} \partial_\mu F^{\mu\nu} = j^\nu \\ \partial_\nu j^\nu = 0 \end{cases}$$

NOETHER

$A_\mu \to A_\mu + \partial_\mu \Lambda$

NOTE: ONE COULD HAVE ADDED

$$\tilde{F}_{\mu\nu} F^{\mu\nu} = \varepsilon_{\alpha\beta\mu\nu} F^{\alpha\beta} F^{\mu\nu}$$

SHOW THAT $F_{\mu\nu} \tilde{F}^{\mu\nu}$ IS A TOTAL DERIVATIVE

$$\pi^\rho = \frac{\partial \mathcal{L}}{\partial \dot{A}_\rho} = F^{\rho 0} \qquad \Rightarrow \pi^0 = 0$$

$\rightarrow$ PROBLEM WITH $\{A^0, \pi^0\}$!

3. QUANTIZATION

- RELATIVISTIC WAVE EQUATIONS
  - KLEIN GORDON
  - DIRAC
  - LAGRANGIANS
- SECOND QUANTIZATION
  - FOCK SPACE
  - RELATIVISTIC BOSON FIELD
  - RELATIVISTIC NORMALISATION

QUANTIZATION

a) PROMOTE p and q

TO $\hat{p}$ and $\hat{q}$

b) ACT ON A HILBERT SPACE

c) $\{ , \} \rightarrow i [,]$

EXAMPLE : COORDINATE REPRESENTATION

$$q^\mu = (t, \vec{x}) \qquad p^\mu = (i\partial_t , -i\vec{\nabla}) = i\partial^\mu$$

$$E = \frac{1}{2m} p^2 \quad \rightarrow \quad i\partial_t \psi = -\frac{1}{2m} \Delta \psi$$

HILBERT SAACE:

SCALAR PRODUCT $\int d^3x\, \varphi^* \psi = \langle \varphi | \psi \rangle$

PROB. INTERP. $\left\{ \begin{array}{l} \text{POSITIVE DEFINITE} \quad \int d^3x\, |\psi|^2 > 0 \\ \text{CONSERVED CURRENT} \end{array} \right.$

$$j_\mu = \left(|\psi|^2 , -\frac{i}{2m} \psi^* \vec{\nabla} \psi + c.c. \right)$$

THE KLEIN-GORDON EQUATION

$$E^2 = \hat{\vec{p}}^{\,2} + m^2$$

$$\Rightarrow (\Box + m^2)\,\phi = 0$$

SCALAR PRODUCT:

$$\partial_\mu (\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*) = 0$$

$$\Rightarrow \quad j^\mu = \phi^* i \partial^\mu \phi - \phi\, i \partial^\mu \phi^*$$

SHOW THIS

$$j^0 = \phi^* i \partial_t \phi - \phi\, i \partial_t \phi^*$$

NOT POSITIVE DEFINITE!

SOLUTIONS:

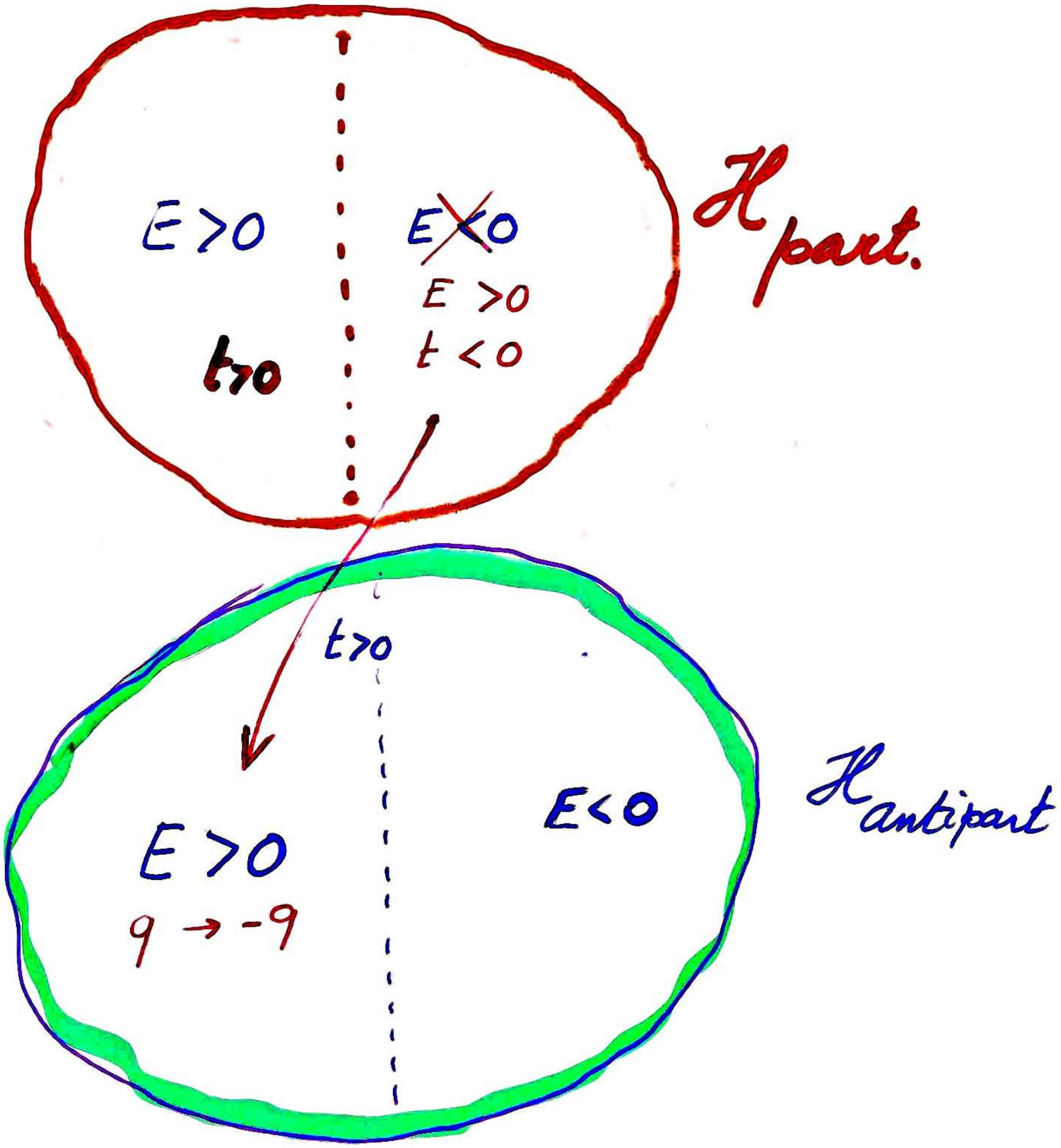
FOR $\vec{p} = 0$:

$$\phi = N e^{\pm imt}$$

$$j^0 = \pm 2m\, |N|^2 \rightarrow \text{CHARGE CURRENT}$$

$\Rightarrow$ NEGATIVE ENERGY STATES HAVE NEGATIVE NORM!

CAN WE LINEARIZE?



$$e^{+i\omega t} = e^{-i\omega(-t)} \qquad \omega = +\sqrt{\vec{p}^2 + m^2}$$

PARTICLE WITH $E<0$ $t>0$

$\Leftrightarrow$ " " $E>0$ $t<0$

$\Leftrightarrow$ ANTIPARTICLE $E>0$ $t>0$

LINEAR FORM 1

$$(\Box + m^2)\,\phi = 0$$

$$\begin{cases} \phi = f + g \\ \frac{1}{m} i\partial_t \phi = f - g \end{cases} \qquad \begin{cases} f = \frac{1}{2}\left(\phi + \frac{i}{m}\partial_t \phi\right) \\ g = \frac{1}{2}\left(\phi - \frac{i}{m}\partial_t \phi\right) \end{cases}$$

$$\Rightarrow \quad \rho = \int (i\phi^* \partial_t \phi - i\phi\partial_t \phi^*)\, d^3x$$

$$= m \int [(f^* + g^*)(f - g) - (f+g)(g^* - f^*)]\, d^3x$$

$$= 2m \int (f^* f - g^* g)\, d^3x$$

$$\begin{cases} i\partial_t f = -\frac{1}{2m}\nabla^2 (f+g) + m f \\ i\partial_t g = \frac{1}{2m}\nabla^2(f+g) - m g \end{cases}$$

$$\frac{1}{2} i\partial_t \phi - \frac{1}{2m}\partial_t^2 \phi = \frac{m}{2}(f-g) - \frac{1}{2m}\left(\nabla^2(f+g) - m^2(f+g)\right)$$

$$= \frac{m}{2}(f - g + f + g) - \frac{1}{2m}\nabla^2(f+g)$$

invent $\quad \psi \equiv \begin{pmatrix} f \\ g \end{pmatrix}$

$$\underline{\rho = 2m\, \psi^\dagger \sigma_3 \psi}$$

$$\underline{i\partial_t \psi = \underbrace{\left[\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\frac{\nabla^2}{2m} + m\sigma_3\right]}_{\hat{H}} \psi}$$

$$\langle \psi | \psi' \rangle = \int d^3x\; \psi^\dagger \sigma_3 \psi'$$

$$\psi(x) = u(p)\, e^{-i p_\mu x^\mu}$$

$$u_+ = N \begin{pmatrix} E + m \\ m - E \end{pmatrix} \qquad u_- = N \begin{pmatrix} m - E \\ m + E \end{pmatrix}$$

LINEARISATION 2 : SPIN

EQUATION LINEAR IN ∂_t AND $\vec{\nabla}$

$$H = \vec{\alpha} \cdot \hat{\vec{p}} + \beta m$$
$$H^2 = |\hat{\vec{p}}|^2 + m^2 = \sum_{ij} (\alpha_i p_i + \beta m)(\alpha_j p_j + \beta m)$$

$$\Rightarrow \quad \begin{cases} \alpha_i^2 = 1 \qquad \beta^2 = 1 \\ \{\alpha_i, \alpha_j\} = \{\beta, \alpha_i\} = 0 \qquad j \neq i \end{cases}$$

α_i, β are hermitian matrices of even dimension of eigenvalues ± 1

H

$\det \alpha_i \alpha_j = (-)^d \det \alpha_j \alpha_i$

$\alpha_i^2 = 1$

Pauli matrices

$$\begin{pmatrix} \cdot & 1 \\ 1 & \cdot \end{pmatrix} \begin{pmatrix} \cdot & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & \cdot \\ \cdot & -1 \end{pmatrix}$$

$\Rightarrow$ Weyl equation : $i \partial_t \psi = \pm \vec{\sigma} \cdot \vec{p}\, \psi$

ok for massless fermions

$d=4$:

$$\vec{\alpha} = \begin{pmatrix} -\vec{\sigma} & \cdot \\ \cdot & \vec{\sigma} \end{pmatrix} \qquad \beta = \begin{pmatrix} \cdot & I \\ I & \cdot \end{pmatrix} \qquad \text{Weyl}$$

$$\vec{\alpha} = \begin{pmatrix} \cdot & \vec{\sigma} \\ \vec{\sigma} & \cdot \end{pmatrix} \qquad \beta = \begin{pmatrix} I & \cdot \\ \cdot & -I \end{pmatrix} \qquad \text{Dirac}$$

Show that these two representations are equivalent

Covariant form of the equation :

$$\gamma_0 \equiv \beta \qquad \vec{\gamma} \equiv \beta \vec{\alpha}$$

$$\Rightarrow \boxed{(i \partial_\mu \gamma^\mu - m)\, \psi = 0}$$

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu}}$$

$$(\gamma^\mu)^\dagger = \gamma_0 \gamma^\mu \gamma_0$$

$$\vec{\gamma}_W = \begin{pmatrix} \cdot & \vec{\sigma} \\ -\vec{\sigma} & \cdot \end{pmatrix}$$

$$\vec{\gamma}_D = \vec{\gamma}_W$$

Current :

$$\boxed{j^\mu = \psi^\dagger \gamma_0 \gamma^\mu \psi}$$

$$\bar{\psi} \equiv \psi^\dagger \gamma_0$$

in particular: $\rho = \psi^\dagger \psi$ positive

NB: FOR QUANTUM FIELDS $\langle \rho \rangle$ CAN BE <0

SOLUTIONS:

$$\omega = \sqrt{\vec{p}^2 + m^2} \qquad \chi_+ \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_- \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\psi = u\, e^{-i p \cdot x}$$

$$\Rightarrow \quad E = +\omega$$

$$u = \begin{pmatrix} \chi \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{\omega + m}\,\chi \end{pmatrix}$$

IN DIRAC REP.

$$E = -\omega$$

$$u = \begin{pmatrix} -\dfrac{\vec{\sigma}\cdot\vec{p}}{\omega+m}\,\chi \\ \chi \end{pmatrix}$$

PROPERTIES:

$$[H, \vec{\Sigma}\cdot\vec{p}] = 0 \qquad \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & \cdot \\ \cdot & \vec{\sigma} \end{pmatrix}$$

$$[H, \vec{L} + \tfrac{1}{2}\vec{\Sigma}] = 0$$

show this.

$\Rightarrow$ TOTAL ANGULAR MOMENTUM

$$\vec{J} = \vec{L} + \frac{\vec{\Sigma}}{2}$$

LORENTZ TRANSFORMATIONS

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$$

$$x_{\mu} x^{\mu} = x'_{\mu} x'^{\mu} \Rightarrow \Lambda_{\mu}{}^{\nu} \Lambda^{\mu}{}_{\tau} = \delta^{\nu}_{\tau}$$

$$(\Lambda^{-1})_{\tau}{}^{\mu} = \Lambda^{\mu}{}_{\tau}$$

$$\partial'_{\mu} = \Lambda_{\mu}{}^{\nu}\, \partial_{\nu}$$

$$\psi \to S\psi = \psi'$$

$$(i\gamma_{\mu}\partial'^{\mu} - m)\psi' = (i\gamma_{\mu}\Lambda^{\mu\nu}\partial_{\nu} - m)\, S\psi$$

$$\gamma^{\mu}\Lambda_{\mu}{}^{\nu} S = S\gamma^{\nu}$$

$$\left.\begin{array}{l} \Lambda = 1 + \varepsilon \\ \boxed{S = I + \Gamma\cdot\varepsilon} \end{array}\right\} \quad (1 + \Gamma\cdot\varepsilon)\,\gamma^{\nu}(1 - \Gamma\cdot\varepsilon) = (1+\varepsilon)\gamma$$

$$[\Gamma\cdot\varepsilon, \gamma_{\nu}] = \varepsilon_{\nu}{}^{\mu}\gamma^{\mu}$$

$$\boxed{\Gamma_{\mu\nu} = \frac{1}{8}[\gamma_{\mu}, \gamma_{\nu}]}$$

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix}}_{\Lambda^{-1} = \Lambda^{T}} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \simeq 1 + \underbrace{\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \theta & \cdot \\ \cdot & -\theta & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}}_{\varepsilon}$$

$$S = 1 + \frac{\theta}{8}\, 2\varepsilon^{12}[\gamma_1, \gamma_2] = 1 + \frac{\theta}{4}\left[\begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}\right]$$

$$= 1 + \frac{\theta}{4}\begin{pmatrix} 2i\sigma_3 & \cdot \\ \cdot & 2i\sigma_3 \end{pmatrix} = 1 + \frac{i\theta}{2}\vec{\Sigma}\cdot\vec{n}$$

$= \text{first order of } \exp\left(\frac{i\theta}{2}\vec{\Sigma}\cdot\vec{n}\right)$!

DISCRETE TRANSFORMATIONS

PARITY $\qquad S_P = \gamma^0$ $\qquad$ SHOW THIS

CHARGE CONJ.

$$\psi_c = C'\psi^* = C(\bar{\psi})^T$$

$$C' = i\gamma_2 = \begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}$$

$$u^{(1,2)}(-p^\mu) = v^{(2,1)}(p^\mu)$$

TIME REVERSAL

$$\psi' = T\psi^*(t)$$

$$T = i\gamma^1\gamma^3$$

$$\psi_c = i\gamma_2\psi^*$$

$$\psi_{cP} = \gamma^0 i\gamma_2\psi^*$$

$$\psi_{cPT} = i\gamma_1\gamma_3\ \gamma_0\ i\gamma_2\psi$$

$$= \gamma_5\psi$$

LORENTZ TRANSFORMATIONS 2

HOW TO COMBINE SPINORS TO MAKE SCALARS, VECTORS...

$\bar{\psi}\psi = \psi^{\dagger}\gamma_0\psi$ SCALAR

$\bar{\psi}\gamma_\mu\psi$ VECTOR

$\bar{\psi}\, i[\gamma_\mu, \gamma_\nu]\,\psi$ TENSOR

$\bar{\psi}\gamma_5\psi$ PSEUDO SCALAR

$\bar{\psi}\gamma_5\gamma_\mu\psi$ PSEUDOVECTOR

$$\underline{\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3}$$

$\gamma_5^{\dagger} = \gamma_5$

$\gamma_5^2 = I$

$\{\gamma_5, \gamma_\mu\} = 0$

$\gamma_5 = \begin{pmatrix} \cdot & I \\ I & 0 \end{pmatrix}$ (Dirac rep.)

$\psi = \begin{pmatrix} a \\ b \end{pmatrix}$ $\gamma^0 = \begin{pmatrix} I & \\ & -I \end{pmatrix}$ $\bar{\psi} = (a^{\dagger}, -b^{\dagger})$

$a^{\dagger}a - b^{\dagger}b$ IS A SCALAR

$\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \Rightarrow (a^{\dagger}a + b^{\dagger}b, +b^{\dagger}\vec{\sigma}a + a^{\dagger}\vec{\sigma}b)$

is A 4-VECTOR

ETC.

LAGRANGIAN

$$\mathcal{L}_D = \bar{\psi}(i\gamma_\mu \partial^\mu - m)\psi$$

$$\frac{\delta \mathcal{L}}{\delta \bar{\psi}} \Rightarrow (i\gamma_\mu \partial^\mu - m)\psi = 0$$

$$\frac{\delta \mathcal{L}}{\delta \partial_\mu \psi} - \frac{\delta \mathcal{L}}{\delta \psi} \Rightarrow \bar{\psi}(i\gamma_\mu \partial^\mu + m) = 0$$

→ VARY ψ AND $\bar{\psi}$ INDEPENDENTLY

WEYL SPINORS

$$\psi = \begin{pmatrix} a \\ b \end{pmatrix}$$

CHIRAL
$$\begin{cases} \psi_R = \frac{1}{2}(1+\gamma_5)\psi = \begin{pmatrix} a \\ 0 \end{pmatrix} \\ \psi_L = \frac{1}{2}(1-\gamma_5)\psi = \begin{pmatrix} 0 \\ b \end{pmatrix} \end{cases}$$

IN WEYL REP.

$$\Rightarrow \bar{\psi}(i\gamma.\partial - m)\psi$$

$$= \bar{\psi}_R \, i\gamma.\partial \psi_R + \bar{\psi}_L \, i\gamma.\partial \psi_L + m(\bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R)$$

$\psi_R \longrightarrow \psi_L$ UNDER P

$\psi_R \sim$ HELICITY STATE IF $M=0$

SECOND QUANTIZATION: BOSONS

HILBERT SPACE

$\mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots$ FOCK SPACE

ISOMORPHISM BETWEEN FOCK SPACE AND AN ∞ NUMBER OF HARMONIC OSCILLATORS

$$H_{h.o.} = \frac{1}{2}(\hat{p}^2 + q^2) \qquad \begin{cases} [p,p] = [q,q] = 0 \\ [q,p] = i \end{cases}$$

$$= \frac{1}{2}\Big[\underbrace{(\hat{p}+iq)}_{a^+}\underbrace{(\hat{p}-iq)}_{a} + 1\Big]$$

$$\begin{cases} [a,a] = [a^+,a^+] = 0 \\ [a,a^+] = 1 \end{cases}$$

$|0\rangle$ = FUNDAMENTAL STATE :

$$a|0\rangle = 0$$

$$H|0\rangle = \frac{1}{2}|0\rangle$$

$\Rightarrow$ $(a^+)^n|0\rangle$ IS AN EIGENSTATE

$$H(a^+)^n|0\rangle = (a^+)^n(H+n)|0\rangle$$

$$= (n+\tfrac{1}{2})(a^+)^n|0\rangle$$

SEVERAL INDEPENDENT OSCILLATORS

$$H = \frac{1}{2}\sum_a \hat{p}_a^2 + q_a^2 \qquad [a_a^+, a_b^+] = [a_a, a_b] = 0$$

$$[a_a, a_b^+] = \delta_{ab}$$

$$|0\rangle = \bigotimes_a |0_a\rangle$$

$$a_a|0\rangle = 0$$

NORMALISATION

$$|A\rangle = \sum \alpha_n \, a_n^{+} \, |0\rangle$$

$$|B\rangle = \sum \beta_n \, a_n^{+} \, |0\rangle$$

$$\langle B|A\rangle = \sum \beta_m^* \alpha_n \, \langle 0| a^m \, a^{+n} |0\rangle$$

$$= \sum \beta_m^* \alpha_n \, \langle 0| a^{m-1} a^{+n} a + a^{m-1} \, n \, a^{+n-1} |0\rangle$$

$$= \sum \beta_m^* \alpha_n \; n! \, \delta_{mn}$$

FOCK STATES :

FREE STATES $|\alpha^1\rangle \dots |\alpha^n\rangle$ $\quad \langle \alpha^a|\alpha^b\rangle = \delta^{ab}$

g PARTICLES $|\psi\rangle = \frac{1}{u!} \sum_{perm} |\alpha_1^a\rangle |\alpha_2^b\rangle \dots |\alpha_u^g\rangle$

$\langle\psi|\psi\rangle = 1$ IF STATES ARE DIFFERENT

$= n!$ IF n STATES ARE IDENTICAL

$\frac{1}{\sqrt{2}}\left(|\alpha^1\rangle|\alpha^1\rangle + |\alpha^1\rangle|\alpha^1\rangle\right) = \sqrt{2}\, |\alpha^1\rangle|\alpha^1\rangle \rightarrow$ NORM 2

$= n_1! \, n_2! \dots n_n!$

$$\boxed{N = \sum a_i^{+} a_i} = \text{TOTAL NUMBER}$$

$$a_\beta \, |\dots n_\beta \dots\rangle = \sqrt{n_\beta} \, |\dots, n_{\beta}-1, \dots\rangle$$

$$a_\beta^{+} \, |\dots n_\beta \dots\rangle = \sqrt{n_\beta+1} \, |\dots, n_{\beta}+1, \dots\rangle$$

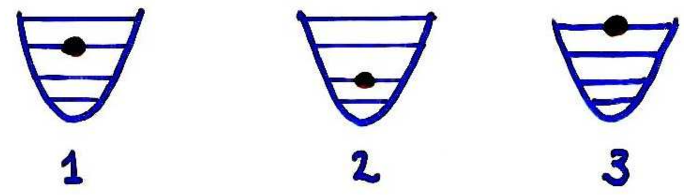
$|\alpha^a\rangle$

$\sqrt{n!}\ \left(|\alpha^a\rangle|\alpha^a\rangle\dots|\alpha^a\rangle\right)$

3 BOSONS IN $|\alpha^3\rangle$
1 BOSON IN $|\alpha^2\rangle$
2 BOSONS IN $|\alpha^1\rangle$

$a_a^+|0\rangle$

$(a_a^+)^n|0\rangle$



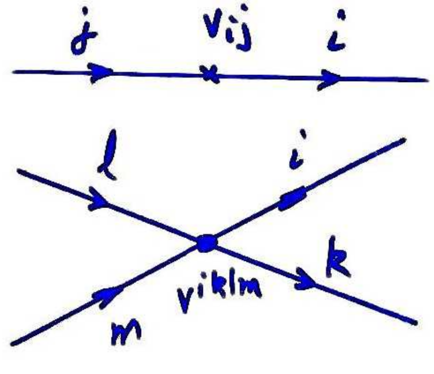
INTERACTION:

$v(\vec{r})$ $\quad \langle\alpha^i|v|\alpha^j\rangle = v^{ij}$

$v(\vec{r}_i-\vec{r}_j)$ $\quad \langle\alpha^i\alpha^k|v|\alpha^l\alpha^m\rangle = v^{iklm}$

$$V = \sum v^{ij}\, a_i^+ a_j$$

$$V = \sum v^{iklm}\, a_i^+ a_k^+ a_l a_m$$



LORENTZ TRANSFORMATIONS

$$x' = \Lambda x + a$$

$$|\phi'\rangle = U(\Lambda, a)\,|\phi\rangle$$

IRREDUCIBLE REPRESENTATION OF SO(3,1)
MASS m, SPIN 0

$$\langle x|U(\Lambda,a)|\phi\rangle = \langle x|\phi'\rangle = \langle \Lambda^{-1}(x-a)|\phi\rangle$$

$$\langle k|\phi'\rangle = e^{ik.a}\langle \Lambda^{-1}k|\phi\rangle$$

N-PARTICLE STATES :

$$\langle k_1 \dots k_N|U(\Lambda,a) = e^{i\sum k_i \cdot a}\langle \Lambda^{-1}k_1 \dots \Lambda^{-1}k_N|$$

$$\langle k_1 \dots k_n|U(\Lambda,a)\,a_k|\phi\rangle$$

$$= \langle \Lambda^{-1}k_1 \dots \Lambda^{-1}k_n|\,a_k\,|\phi\rangle\, e^{i\sum k_i a}$$

$$= \sqrt{n_k+1}\;\langle k, \Lambda^{-1}k_1 \dots \Lambda^{-1}k_n|\phi\rangle\, e^{i\sum k_i a}$$

$$= \langle k_1 \dots k_n|e^{-i\Lambda k.a}\;a(\Lambda k)\;U(\Lambda,a)\,|\phi\rangle$$

$$= \langle k_1 \dots k_n|\,U(\Lambda,a)\,U^{-1}(\Lambda,a)\,a(\Lambda,k)\,U(\Lambda,a)\,|\phi\rangle\, e^{-i\Lambda k.a}$$

$$\Rightarrow \boxed{\begin{cases} U(\Lambda,a)\,a_k\,U(\Lambda,a)^{-1} = e^{-i\Lambda k.a}\,a(\Lambda k) \\ U(\Lambda,a)\,a_k^{+}\,U(\Lambda,a)^{-1} = e^{i\Lambda k.a}\,a^{+}(\Lambda k) \end{cases}}$$

FIELDS

CAN WE DEFINE OPERATORS THAT TRANSFORM LIKE SCALARS, I.E.

$$U \phi(x) U^{-1} = \phi(\Lambda x + a) \quad ?$$

$$\boxed{\phi^{(+)}(x) = \int_{k_0>0} \frac{d^3k}{k_0} \frac{1}{\sqrt{2(2\pi)^3}} e^{-ik.x} a_{\vec{k}}}$$

$$(\Box + m^2)\, \phi^+(x) = 0 \qquad k_0 = \sqrt{\vec{k}^2 + m^2}$$

$$\boxed{\phi^-(x) = \int_{k_0>0} \frac{d^3k}{k_0} \frac{1}{\sqrt{2(2\pi)^3}} e^{+ik.x} a^+_{\vec{k}}}$$

$$U(\Lambda,a)\, \phi^+(x)\, U(\Lambda,a)^{-1} = \int_+ d\Omega(k)\, e^{-ik.x}\, U a_{\vec{k}} U^{-1}$$

$$= \int_+ d\Omega(k)\, e^{-i\Lambda k.\Lambda x}\, e^{-i\Lambda k.a}\, a_{\Lambda\vec{k}}$$

$$= \int_+ d\Omega(k')\, e^{-ik'.(\Lambda x + a)}\, a_{\vec{k}'}$$

$$= \phi^+(\Lambda x + a)$$

SIMILARLY FOR ϕ^-

HERMITIAN FIELD

$$\phi(x) = \phi^+(x) + \phi^-(x)$$

$$(\Box + m^2)\,\phi(x) = 0$$

$$[\phi(x), \phi(x')] = \frac{1}{2(2\pi)^3} \int_{k_0>0} \frac{d^3k}{k_0} \left(e^{-ik(x-x')} - e^{ik\cdot(x-x')}\right)$$

prove this $\rightarrow$

$$\begin{cases} = \dfrac{+i}{(2\pi)^3} \displaystyle\int_{k_0>0} e^{ik(x-x')} \dfrac{d^3k}{k_0} \sin k_0(x_0 - x_0') \\ = 0 \quad \text{IF} \quad x_0 = x_0' \end{cases}$$

$$\equiv \Delta(x - x')$$

$$\Delta(\vec{x}, t) = \frac{1}{4\pi} \frac{i}{r} \partial_r \, F(r, t)$$

$$F(r,t) = \begin{cases} J_0(m\sqrt{t^2 - r^2}) & t > r \\ 0 & -r < t < r \\ -J_0(m\sqrt{t^2 - r^2}) & t < -r \end{cases}$$

$$\Delta(x) = -\frac{i}{2\pi} \varepsilon(t) \left\{ \delta(x^2) - \frac{m^2}{2} \theta(x^2) \frac{J_1(m\sqrt{x^2})}{m\sqrt{x^2}} \right\}$$

CONJUGATE FIELD

$$\pi(x) = \partial_t \phi(x)$$

$$= -\frac{i}{\sqrt{2(2\pi)^3}} \int_{k_0>0} d^3k \left(e^{-ik.x} a_{\vec{k}} - e^{ik.x} a_{\vec{k}}^{+} \right)$$

$$\boxed{\left[\pi(x), \phi(x')\right]_{t=t'} = -i\,\delta^{(3)}(\vec{x}-\vec{x}\,')}$$

prove this

$\Rightarrow$ WE RECOVER THE LAGRANGIAN COMMUTATORS!

RELATIVISTIC NORMALISATION

$$|\vec{p}\rangle = C\, a^{\dagger}_{\vec{p}} |0\rangle$$

$$\langle \vec{p} | \vec{q} \rangle = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})\; C^2$$

$$\xrightarrow{\Lambda} \quad |p'\rangle$$

$$\langle p' | q' \rangle = (2\pi)^3 \delta^{(3)}(\vec{p}\,'-\vec{q}\,')\, C^2$$

$$\int d^3p'\; \delta^{(3)}(\vec{p}\,'-\vec{q}\,') = \int \frac{p_0'}{p_0} \delta^{(3)}(\vec{p}\,'-\vec{q}\,')\, d^3p$$

$\dfrac{d^3p}{p_0}$ invariant $\rightarrow$ replace d^3p with this

$$\Rightarrow p^0 \delta^{(3)}(\vec{p}-\vec{q}) = p_0' \delta^{(3)}(\vec{p}\,'-\vec{q}\,')$$

ABSORB THIS IN THE NORMALISATION

$$|p\rangle = \sqrt{2E_p}\; a^{\dagger}_{\vec{p}} |0\rangle \qquad [|p\rangle = GeV^{-1}]$$

$$\langle p | q \rangle = (2\pi)^3\; 2E_p\, \delta^{(3)}(\vec{p}-\vec{q}) \qquad [\langle p|q\rangle] = GeV^{-2}$$

$$[a_{\vec{p}}, a^{\dagger}_{\vec{p}'}] = (2\pi)^3 \cdot \delta^{(3)}(\vec{p}-\vec{p}\,') \qquad [a] = GeV^{-3/2}$$

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} + a^{\dagger}_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}} \right)$$

$$[\phi] = GeV$$

RECIPE

$$S = \int d^4x \, \mathcal{L}$$

- SYMETRIES LEAVE S INVARIANT
- S EXTREMUM

  $\longrightarrow$ EULER - LAGRANGE FOR FIELDS

- CONJUGATE MOMENTA π

EQUAL TIMES
$$\begin{cases} [\pi , \phi] = i\,\delta^{(3)} \\ [\pi , \pi] = [\phi , \phi] = 0 \end{cases}$$

- EXPAND IN TERMS OF EIGEN FUNCTIONS

  $\rightarrow$ COEFFICIENTS ARE CREATION + ANNIHILATION OPERATORS

  $\rightarrow$ HILBERT SPACE KNOWN

- EVOLVE OPERATORS USING H

$$i\partial_t O = [H, O]$$

- *ONLY ONE FIELD IN ALL SPACETIME*

  $\rightarrow$ *PARTICLES ARE IDENTICAL*