Exercises on QCD

J.R. Cudell

Sept. 2004

1 Lecture 1: flavour, colour, lagrangian

1. In the naïve quark model, write down the wave function of pions in terms of quark and antiquark wave functions, taking into account both spin and the $SU(2)_F$ isospin flavour symmetry. What is the effect of colour on these wave functions? What is the colour part the wave function of a baryon?

2. Construct the original Yang-Mills lagrangian by making the isospin symmetry local. Discuss the properties of the gauge boson. Could it correspond to a physical particle?

3. Derive the classical field equations from the QCD lagrangian, in the case where there are only gluons. Compare with the corresponding Maxwell equations.

4. At the group level, the gauge transformations of $SU(3)_c$ are as follows:

$$q(x) \to U(x)q(x)$$

$$\mathbf{A}_\mu(x) \equiv \sum_a \frac{\lambda_a}{2} A^a_\mu(x) \to U(x)\mathbf{A}_\mu U^\dagger(x) + \frac{i}{g}(\partial_\mu U(x))U^\dagger(x)$$

a) Show that the infinitesimal form agrees with that seen in lecture;
b) Show that the covariant derivative $D_\mu = \partial_\mu + ig\mathbf{A}_\mu$ transforms as

$$D_\mu q(x) \to U(x) D_\mu q(x)$$

c) Show that the gluon field-strength

$$\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + ig[\mathbf{A}_\mu, \mathbf{A}_\nu]$$

transforms as

$$\mathbf{F}_{\mu\nu} \to U(x)\mathbf{F}_{\mu\nu}U^\dagger(x)$$

d) Show that the QCD lagrangian is invariant w.r.t. these group transformations.

Discussion

a. Sketch the strong potential as a function of distance. What does one mean when one says that strong interactions are weak at short distances, although $V(r) \sim 1/r$?

b. What is the relations between the flavour symmetry $SU(3)_F$ and the colour symmetry $SU(3)_C$?

c. Why does one assume that the colour group is unitary?

d. We assumed that matter comes under triplets of $SU(3)$ and gauge bosons are either singlets or octets. Are there other representations of $SU(3)$?

2 Lecture 2: Feynman rules, colour flow

1. Calculate the following sums, from the algebra and from colour-flow diagrams:

$$\sum_{ab} f_{abc} f_{abd}$$

$$Tr(T_a T_b)$$

$$Tr(T_a T_b T_a)$$

where $T_a = \lambda_a/2$.

2. Give the rules for colour-flow diagrams in the limit $N_c \to \infty$. Draw the leading diagrams for

$$q\bar{q} \to gg$$

$$q\bar{q} \to q\bar{q}$$

$$gg \to q\bar{q}g$$

3. Show that the gluon triple vertex is necessary to make the process $q\bar{q} \to gg$ gauge invariant. Compare with the QED case.

> Hint: first treat the QED case, writing the amplitude in terms of quark u and/or v spinors. Then show that a commutator of λ_a appears in the QCD case. Express it using the algebra, and show that the triple-vertex has precisely the structure needed to cancel it.

Discussion

a. One could add to the lagrangian of QCD a term $\epsilon^{\mu\nu\rho\sigma} Tr(\mathbf{F}_{\mu\nu}\mathbf{F}_{\rho\sigma})$. What would be its consequences? How would it affect the Feynman rules?

b. The running coupling has a pole at Λ_{QCD}. If one took that pole seriously, what would be its consequences in *e.g.* $q\bar{q} \to q\bar{q}$? Compare with the case in which a resonance is produced.

3 Lecture 3: DIS and jets

1. What relations does momentum conservation impose on the splitting functions?

2. In the case of scattering on a single quark, give the expression of the form factors F_1 and F_2. Give the expressions for the interaction cross section $\sigma_{tot}(\gamma^* q \to X)$ for longitudinal and for transverse photons.

3. For low x, find the dominant splitting functions, and show that gluons dominate other partons. Solve the DGLAP equation in that limit as a series in $\log(1/x)$. Assume $g(x) = G/x$ or $g(x) = G/x^{1+\epsilon}$ at $Q^2 = Q_0^2$.

4. Give the values of thrust for back-to-back jets and for spherical events.

5. Another variable used to define jets is spherocity:

$$S = (4/\pi)^2 \min_n \left(\frac{\sum_i |\vec{p}_i \times \vec{n}|}{\sum_i |\vec{p}_i|} \right)$$

Is it infrared stable? What are its values for back-to-back jets and for spherical events?

4 Lecture 4: small x and soft physics

1. The lowest order BFKL diagram is quark-quark scattering via the exchange of two gluons. Write down the diagrams, and calculate their imaginary part in the high-s (small x) limit.

2. Left-handed and right-handed quarks are given by

$$q_L = \frac{1-\gamma_5}{2}, \; q_R = \frac{1+\gamma_5}{2}$$

a) Rewrite the QCD lagrangian for n_f massless quark flavours in terms of q_L and q_R. What is the symmetry of the corresponding lagrangian?
b) The symmetry is spontaneously broken by the quark condensate $< \Omega| : \bar{q}q : |\Omega > \approx (250 \text{ MeV})^3$. Verify that this condensate does break the symmetry. Is there a symmetry that survives?
c) Show that the remaining symmetry is broken by a mass term for the quarks.

3. Gribov has proposed the following form for the gluon propagator in the Landau gauge:

$$D^{ab}_{\mu\nu}(q^2) = \delta^{ab}\left(g_\mu\nu - \frac{q_\mu q_\nu}{q^2}\right) \times \frac{q^2}{q^4+m^4}$$

Would the gluons described by this propagator be confined? What would be their effective mass? Does such a propagator allow Wick rotations?

4. The plaquette action seen in lecture is given by

$$S = \beta \sum_{x,\mu,\nu} (1 - 1/3 P_{\mu\nu}(x))$$

with

$$P_{\mu\nu} = Tr\left(1 - U_\mu(x)U_\nu(x+ae^\mu)U^\dagger_\mu(x+ae^\nu)U^\dagger_\nu(x)\right)$$

and $U_\mu(x)$ the parallel transporter $U(x+ae^\mu, x)$. Show that it reduces to the Yang-Mills action when $a \to 0$.
Hint: work first on the plaquette at the origin, in the gauge $A_\nu(x) = 0$, $A_\mu(x^\nu = 0) = 0$.